SUPPLEMENTARY APPENDIX

Strategy-proof and Efficient Mediation: An Ordinal Market Design Approach

by

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Proof of Theorem 3

Theorem 3 (Visual Characterization). *The following statements are equivalent for the lower half of the mediation rule f, corresponding to type profiles in which a mutually acceptable alternative from the main issue exists;*

(*i*) *f is a logrolling rule.*

(*ii*) *The triangle $\triangle_{m,1}$ has a rectangular partition such that f assigns a unique bundle from the set of logrolling bundles B^t to each rectangle in this partition.*[1]

Proof. We start with $(i) \Rightarrow (ii)$. Let $f_{r_1,r_1} = (x_{r_1}, t(x_{r_1})) \in B^t$ where $x_{r_1} = \underset{X}{\boldsymbol{max}} \trianglerighteq$ and $1 \leq r_1 \leq m$. Because the mediation rule f always chooses the alternative in the first issue that maximizes $\trianglerighteq$, all the entries on row r_1 to the left of entry f_{r_1,r_1}, all the entries on column r_1 below entry f_{r_1,r_1}, and all the entries in between must fill up with bundle $(x_{r_1}, t(x_{r_1}))$ because x_{r_1} has the highest rank over X. Thus, the rectangle $\square^{r_1}_{m,1}$ fills up with $(x_{r_1}, t(x_{r_1}))$.

Let $\square^{r_1}_{m,1}$ be the first element of the rectangular partition of $\triangle_{m,1}$. Note that, when $m \geq 3$, the so-far-unfilled $\triangle_{m,1} \backslash \square^{r_1}_{m,1}$ consists of at least one triangle (if $r_1 \in \{1, m\}$) and at most two triangles (if $r_1 \notin \{1, m\}$).

Next, take an arbitrary triangle $\triangle_{s,r} \in \triangle_{m,1} \backslash \square^{r_1}_{m,1}$. Note that either $s = r_1$ and $r = 1$, or $s = m$ and $r = r_1 + 1$. Let $f_{r_2,r_2} = (x_{r_2}, t(x_{r_2})) \in B^t$ with $r_2 \neq r_1$ denote the logrolling bundle on the hypotenuse of $\triangle_{s,r}$ that satisfies $x_{r_2} = \underset{X_{sr}}{\boldsymbol{max}} \trianglerighteq$. Once again, starting from the hypotenuse of $\triangle_{s,r}$ all the so-far-unfilled entries on row r_2 to the left of entry f_{r_2,r_2}, all the so-far-unfilled entries on column r_2 below entry f_{r_2,r_2}, and all entries in between must fill up with bundle $(x_{r_2}, t(x_{r_2}))$ because x_{r_2} has the highest rank among X_{sr}. Thus, let $\square^{r_2}_{s,r}$ denote the second element of the rectangular partition of $\triangle_{m,1}$.

Note that the so-far-unfilled set $\triangle_{m,1} \backslash \left\{\square^{r_1}_{m,1} \cup \square^{r_2}_{s,r}\right\}$ consists of at least one triangle. Iterate this reasoning and at each step pick a triangle from the so-far-unfilled subset of $\triangle_{m,1}$ and fill its corresponding rectangle with the bundle whose first component has the highest precedence with respect to $\trianglerighteq$. By the finiteness of the problem, the rectangular partition is obtained in m steps.

[1] More formally, for any $\square$ in the partition of $\triangle_{m,1}$ and any bundles $b, b' \in \square$, $b = b'$; but for any distinct pair $\square, \square'$ in the partition of $\triangle_{m,1}$, $(x, y) \in \square$ and $(x', y') \in \square'$ implies $x \neq x'$ and $y \neq y'$.

Now we show $(ii) \Rightarrow (i)$. Consider a rectangular partition $\mathcal{P}^1$ of $\triangle_{m,1}(\equiv \triangle^1)$. Let $\square^{r_1} \subset \triangle^1$ be the rectangle that includes the entry at the bottom left corner of triangle $\triangle^1$, i.e., $f_{m,1}$. We construct the precedence order $\trianglerighteq$ as follows: Let $f^X_{m,1} = x_{r_1}$ have the higher precedence rank than any other alternative in X, i.e., $x_{r_1} \trianglerighteq x$ for all $x \in X$. Next consider $\triangle^1 \backslash \square^{r_1}$ which has a triangular partition $\mathcal{P}^2$ that consists of at most two triangles.

Take an arbitrary triangle $\triangle^2 \in \mathcal{P}^2$ and let $\square^{r_2} \subset \triangle^2$ denote the rectangle that includes the entry at the bottom left corner of triangle $\triangle^2$, say $(x_{r_2}, t(x_{r_2}))$. Let x_{r_2} have a higher precedence rank than any other alternative in X that appears on the hypotenuse of $\triangle^2$. Namely, if $r_2 < r_1$, then $x_{r_2} \trianglerighteq f^X_{k,k}$ for all $k \in \{1, ..., r_2 - 1, r_2 + 1, ..., r_1 - 1\}$, and if $r_2 > r_1$, then $x_{r_2} \trianglerighteq f^X_{k,k}$ for all $k \in \{r_1 + 1, ..., r_2 - 1, r_2 + 1, ..., m\}$.

Iterate in this fashion by considering an arbitrary triangle from the remaining partition $\triangle^1 \backslash \{\square^{r_1}, \square^{r_2}\}$. At the end of this finite procedure (consisting of exactly m steps), we obtain a transitive, antisymmetric but possibly incomplete strict precedence order $\trianglerighteq$ on B^t. Moreover, by construction we have $f_{\ell,j} = \left(x^*_{X_{j\ell}}, t\left(x^*_{X_{j\ell}}\right)\right)$ where $x^*_{X_{j\ell}} = \underset{X_{j\ell}}{\boldsymbol{max}} \trianglerighteq$ for all $1 \leq j \leq \ell \leq m$. This completes the proof. $\square$

Proof of Theorem 4

Theorem 4. *A mediation rule minimizes rank variance within the class of logrolling rules if and only if it is a constrained shortlisting rule.*

Proof. Clearly, a CS rule belongs to the logrolling rules family. Fix the set of logrolling bundles $B^t = \{(x, t(x))|x \in X\}$ and the family of logrolling rules whose range is $B^t \cup \{(o_X, y)\}$ for some $y \in Y$. Let $b_j = (x_j, t(x_j)) \in B^t$ denote a logrolling bundle. To see that the rank variance of a CS rule is lower than any other member of the logrolling rules family, we simply consider two cases about the number of possible alternatives. First, when m is odd, $var(b_k) = (m+1)^2$. For any $b_{k-j}, b_{k+j} \in B^t$ with $j < k$, we have $var(b_{k-j}) = var(b_{k+j}) = 2(\frac{(m+1)}{2} - j)^2 + 2(\frac{(m+1)}{2} + j)^2 = (m+1)^2 + 4j^2$. Thus, $var(b_k) < var(b)$ for any $b \in B^t \setminus \{b_k\}$.

Since any member of the logrolling rules family must pick an element of B^t whenever there is a mutually acceptable alternative in issue X (by Theorem 1), minimization of rank variance requires that $x_k \trianglerighteq x$ for any $x \in X$. Also observe that $var(b_k) < var(b_{k-1}) < \ldots < var(b_1)$ and $var(b_k) < var(b_{k+1}) < \ldots < var(b_m)$. Thus, minimization of rank variance subsequently requires that $x_{k-1} \trianglerighteq \ldots \trianglerighteq x_1$ and $x_{k+1} \trianglerighteq \ldots \trianglerighteq x_m$. By Theorem 1 the outcome for issue X is fixed to o_X whenever there is no mutually acceptable alternative in this issue. Therefore, (o_X, y_k) is the rank-variance-minimizing bundle. Note that when m is odd, rank variance of the unique CS rule is strictly less than any other member of the logrolling rules family.

On the other hand, when m is even, $var(b_{\overline{k}}) = var(b_{\underline{k}}) = \frac{1}{2}(m^2 + (m+2)^2)$. For any $b_{\underline{k}-j}, b_{\overline{k}+j} \in B^t$ with $j < k$, we have $var(b_{\underline{k}-j}) = var(b_{\overline{k}+j}) = 2(\frac{m}{2} - j)^2 + 2(\frac{(m+2)}{2} + j)^2 = \frac{1}{2}(m^2 + (m+2)^2) + 4j^2$. Hence, $var(b_{\overline{k}}) = var(b_{\underline{k}}) < var(b)$ for any $b \in B^t \backslash \{b_{\overline{k}}, b_{\underline{k}}\}$. Note that we also have $var(b_{\underline{k}}) = var(b_{\overline{k}}) < var(b_{\underline{k}-1}) < \ldots < var(b_1)$ and $var(b_{\underline{k}}) =$

$var(b_{\overline{k}}) < (b_{\overline{k}+1}) < \ldots < var(b_m)$. Then, minimization of rank variance subsequently requires that either $x_{\overline{k}} \trianglerighteq x_{\underline{k}}$ or $x_{\underline{k}} \trianglerighteq x_{\overline{k}}$ together with $x_{\underline{k}-1} \trianglerighteq \ldots \trianglerighteq x_1$ and $x_{\overline{k}+1} \trianglerighteq \ldots \trianglerighteq x_m$. By Theorem 1 the outcome for issue X is o_X and both $(o_X, y_{\overline{k}})$ and $(o_X, y_{\underline{k}})$ are rank-variance-minimizing bundles. Consequently, any one of the four types of CS rules are rank variance minimizing. Note that when m is even, rank variance of a CS rule is weakly less than any other member of the logrolling rules family. $\square$

Symmetric Treatment of the Outside Options

We now consider a relaxation of the assumption that $y\ \theta_i^Y o_Y$ for all $i \in \mathrm{N}$ and $y \in Y$. Namely, we suppose that the ranking of each outside option is negotiators' private informations. Let $\Theta_i = \Theta_i^X \times \Theta_i^Y$ denote the set of all **types** for negotiator i, and $\Theta = \Theta_1 \times \Theta_2$ be the set of all type profiles. We now also need to adjust the regularity assumption concerning the negotiators' preferences over bundles. Specifically, we need to modify the bargaining ranges conditions since both issues can now potentially have unacceptable alternatives. For any type θ_i, we let $B(\theta_i) = \{(x, y) \in X \times Y \mid x\ \theta_i^X\ o_X \text{ and } y\ \theta_i^Y\ o_Y\}$ to be the set of all acceptable bundles.

Definition S.1. *Under the symmetric treatment of the outside options, a preference map Λ is regular if the following hold for all $i \in N$ and $\theta_i \in \Theta_i$:*

i. [Monotonicity] For any $x, x' \in \overline{X}$ and $y, y' \in \overline{Y}$ with $(x, y) \neq (x', y')$,

$$(x, y)\ P_i\ (x', y') \text{ for all } R_i \in \Lambda(\theta_i) \text{ whenever } x\ \theta_i^X\ x' \text{ and } y\ \theta_i^Y\ y'.$$

ii. [Consistency] For any $\theta_i' \in \Theta_i$ with $B(\theta_i) \subseteq B(\theta_i')$,

$$\Lambda(\theta_i')|_{B(\theta_i)} = \Lambda(\theta_i)|_{B(\theta_i)}.$$

iii. [Bargaining ranges (BR)] $(o_X, o_Y)\ P_i\ (x, y)$ for all $R_i \in \Lambda(\theta_i)$ whenever $o_X\ \theta_i^X\ x$ or $o_Y\ \theta_i^Y\ y$.

Proposition 1. *Under the symmetric treatment of the outside options, there is no mediation rule f that is strategy-proof, individually rational, and efficient.*

Proof. Consider the preference profile $(\theta_1, \theta_2) = (\theta_1^{x_m}, \theta_1^{y_m}, \theta_2^{x_1}, \theta_2^{y_1})$. That is, both negotiators find all alternatives acceptable. Let $(x, y) = f(\theta_1, \theta_2)$. Because negotiators' preferences over alternatives are diametrically opposed for each single issue, there is at least one negotiator $i \in N$ and an issue for which negotiator i does not get her top alternative for that issue. Suppose, without loss of generality, that this negotiator is 1 and the issue is X: that is, $x \neq x_1$. Consider the new profile where only negotiator 1's preferences are different, $(\theta_1', \theta_2) = (\theta_1^{x_1}, \theta_1^{y_1}, \theta_2^{x_1}, \theta_2^{y_1})$. Individual rationality of f and bargaining ranges property imply that $f(\theta_1', \theta_2) = (x_1, y_1)$ because all other bundles are unacceptable for type $(\theta_1^{x_1}, \theta_1^{y_1})$.

To conclude, we already know that $f(\theta_1, \theta_2) = (x, y)$ and $x \neq x_1$, which implies $x_1\ \theta_1^{x_m}\ x$. Because y_1 is negotiator 1's best alternative in issue Y, either $y = y_1$ or $y_1\ \theta_1^{y_1}\ y$ is true. In either case, Monotonicity and transitivity of preferences imply $(x_1, y_1)\ P_1\ (x, y)$ for all $R_1 \in \Lambda(\theta_1)$. Thus, by misrepresenting her preferences at profile (θ_1, θ_2), negotiator 1 can achieve the bundle (x_1, y_1) that is strictly better than (x, y) for all $R_1 \in \Lambda(\theta_1)$, contradicting the strategy-proofness of f. $\square$

Mediation with Continuum of Alternatives

We extend the characterization of the class of logrolling rules to a continuous analogue of our model.[2] Suppose now that the issues X and Y are two closed and convex intervals of the real line. The outside options, o_X and o_Y, may or may not be the elements of these sets. We assume, without loss of generality, that $X = Y = [0, 1]$, with the interpretation that the negotiators aim to divide a unit surplus in each issue. To keep the notation consistent with the main text, let a bundle $b = (x, y)$ indicate what negotiator 2 gets in the two issues, i.e., negotiator 2 gets $x \in X$ and $y \in Y$, and thus, negotiator 1 gets the remaining $1 - x$ and $1 - y$ in issues X and Y, respectively. Agents having diametrically opposed preferences on each issue means that for any issue $Z \in \{X, Y\}$ and two alternatives $z, z' \in Z$, negotiator 1 (respectively 2) prefers z to z' whenever $z < z'$ (respectively $z > z'$). The value/ranking of the outside option o_X in issue X is each negotiator's private information. However, the value/ranking of the outside option o_Y in issue Y is common knowledge, and both negotiators prefer all $y \in Y$ to o_Y.

For any $\ell \in [0, 1]$, type ℓ of negotiator 1 (respectively 2), denoted by θ_1^ℓ (respectively θ_2^ℓ), prefers the outside option o_X to all alternatives $k \in [0, 1]$ with $\ell < k$ (respectively $\ell > k$).[3] In parallel with the discrete case, we denote a mediation rule by $f = [f_{\ell,j}]_{(\ell,j)\in[0,1]^2}$ where $f_{\ell,j} = f(\theta_1^\ell, \theta_2^j)$ for all $0 \le \ell, j \le 1$.[4] The negotiators have no mutually acceptable alternative in issue X at type profile $(\theta_1^\ell, \theta_2^j)$ when $\ell < j$. The set of mutually acceptable alternatives in issue X is $A(\theta_1^\ell, \theta_2^j) = [j, \ell] = X_{j\ell}$ whenever $\ell \ge j$. We use Θ_i as the set of all types of negotiator i and $\theta_i \in \Theta_i$ as the generic element whenever there is no need to specify the type's least acceptable alternative. The regularity and quid pro quo conditions in the main text directly apply here. The same is true for the definitions of strategy-proofness, efficiency, and individual rationality. In this framework an injective and order-reversing function $t : X \to Y$ corresponds to a strictly decreasing function. When (X, d) is a metric space with a proper metric d, the connected set $X_{j\ell}$ with $\ell \ge j$ is a nonempty, compact, and convex subset of X. Each t function in Figure 1 (in fact any such decreasing function) generates a set of logrolling bundles, B^t.

[2] Matsuo (1989) shows that it is possible to overcome the impossibility in the bilateral exchange model of Myerson and Satterthwaite (1983) by restricting to a finite number of types. This section also shows that the possibility results in our main model are not driven by the finiteness of the number of types.

[3] In other words, $1 - \ell$ (respectively ℓ) is the least acceptable amount of X for type θ_1^ℓ (respectively θ_2^ℓ). Therefore, all k with $\ell \ge k$ (respectively $\ell \le k$) are deemed acceptable by type θ_1^ℓ of negotiator 1 (respectively, by type θ_2^ℓ of negotiator 2).

[4] We assume, without loss of generality, that each negotiator has at least one acceptable alternative. Therefore, there is no type profile where a negotiator deems all alternatives unacceptable.

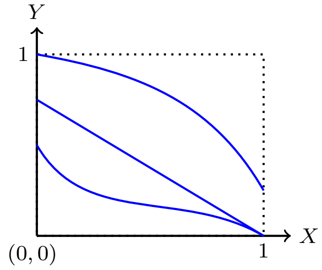

Figure 1: A possible set of logrolling bundles

Definition S.2. *A linear order* $\succeq$ *on* X *is said to be* ***quasi upper-semicontinuous over*** $X_{j\ell}$ *with* $\ell \geq j$ *if for all* $x, x' \in X_{j\ell}$ *with* $x \neq x'$, $x \succeq x'$ *implies that there exists an alternative* $x'' \in X_{j\ell}$ *and a neighborhood* $\mathcal{N}(x')$ *of* x' *such that* $x'' \succeq x'''$ *for all* $x''' \in \mathcal{N}(x') \cap X_{j\ell}$.[5]

The binary relation $\succeq$ is quasi upper-semicontinuous if it is quasi upper-semicontinuous over all compact subsets $X_{j\ell}$ of X. Theorem 1 of Tian and Zhoub (1995) proves that quasi upper-semicontinuity is both necessary and sufficient for $\succeq$ to attain its maximum over all compact subsets $X_{j\ell}$ of X. Therefore, the analogous characterization in the continuous model reads as follows.

Theorem S.1. *Suppose that the preference map* Λ *satisfies quid pro quo. The mediation rule* f *is strategy-proof, efficient, and individually rational if and only if there exists an alternative* $y \in Y$, *a strictly decreasing function* $t : X \to Y$ *and a linear order* $\succeq_t$ *on* X, *which are induced by the preference map* Λ, *such that* $f = f^{\succeq_t}$; *namely*

$$f_{\ell,j} = \begin{cases} \left(x^*_{X_{j\ell}}, t\left(x^*_{X_{j\ell}}\right)\right), & \text{if } j \leq \ell \\ (o_X, y), & \text{otherwise} \end{cases}$$

where $x^*_{X_{j\ell}} = \max\limits_{X_{j\ell}} \succeq_t$ *is well-defined.*

Analogous to the discrete case, we use the following continuously indexed matrix to describe a mediation rule $f^{\succeq_t}$ (see Figure 2). The rows (i.e., the vertical axis) correspond to the types of negotiator 1 and columns (i.e., the horizontal axis) indicate all possible types of negotiator 2. Each point on the main diagonal represents a logrolling bundle for the mediation rule that is described in Theorem S.1, and each logrolling bundle appears only once on this diagonal. The bundle b, for example, represents the value of $f^{\succeq_t}$ when the types of negotiator 1 and 2 are θ_1^ℓ and θ_2^j, respectively. When the type profile is (θ_1^1, θ_2^1), negotiator 1 finds all alternatives acceptable and negotiator 2 deems all alternatives except 1 unacceptable.

If, for example, $t(x) = 1-x$, then the set of logrolling bundles is $B^t = \{(x, 1-x) | x \in [0,1]\}$. Then the only mutually acceptable logrolling bundle is $(1,0)$ at type profile (θ_1^1, θ_2^1). The set of all acceptable logrolling bundles for type θ_1^ℓ of negotiator 1 is denoted by $B^t_{1,\ell}$, which consists of all the logrolling bundles on the upper portion of the main diagonal, starting from the top-left corner bundle, $(0,1)$, and goes all the way down to the

[5]This is Definition 2 in Tian and Zhoub (1995).

bundle $(\ell, 1-\ell)$. That is, $B^t_{1,\ell} = \{(k, 1-k) \in B^t \mid 0 \le k \le \ell\}$. Similarly, the set of all acceptable logrolling bundles for type θ_2^j of negotiator 2 is represented by $B^t_{2,j}$ and consists of all the bundles on the lower portion of the main diagonal, i.e., all bundles from $(j, 1-j)$ to $(1, 0)$. Namely, $B^t_{2,j} = \{(k, 1-k) \in B^t \mid j \le k \le 1\}$. Thus, the set of mutually acceptable logrolling bundles at the type profile $(\theta_1^\ell, \theta_2^j)$ is the intersection of these two sets, i.e., $B^t_{\ell j} = B^t_{1,\ell} \cap B^t_{2,j}$. Theorem S.1 says that bundle $b = f^{\succeq_t}_{\ell,j}$ is the logrolling bundle that maximizes $\succeq_t$ over the set $B^t_{\ell j}$ (see Figure 2). Such a maximal bundle is always unique because $\succeq_t$ is antisymmetric.

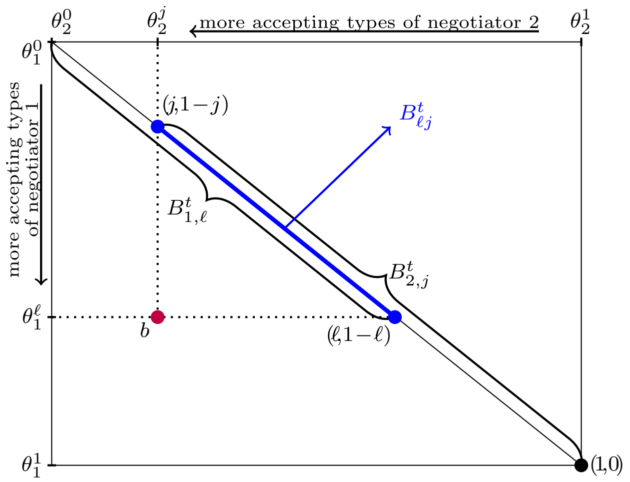

Figure 2: Adjacent rules in the continuous case when $t(x) = 1 - x$

Proof of Theorem S.1:

Proof of 'if': The same arguments in the proof of Theorem 2 suffice to verify that the mediation rule described in Theorem S.1 is individually rational and efficient. Lemma 4 also holds in the continuous case. The proof of part (i) of Lemma 4 is straightforward; given the location of a logrolling bundle $a = (x_a, t(x_a))$ on the main diagonal, $f^{\succeq_t}_{\ell,j}$ can be a only if $a \in B^t_{\ell j}$, and so, a can never appear outside of its value region $V(a)$. To prove part (ii), let $f^{\succeq_t}_{\ell,j} = a$, $f_{s,r} = b = (x_b, t(x_b))$ and $a \neq b$. Also, suppose for a contradiction that $a, b \in V(a) \cap V(b)$. Therefore, we have $a, b \in B^t_{\ell j} \cap B^t_{sr}$. The alternative x_a beats x_b with respect to $\succeq_t$ because a wins over $B^t_{\ell j}$, i.e., $f^{\succeq_t}_{\ell,j} = a$. Likewise, x_b beats x_a with respect to $\succeq_t$ because b wins over B^t_{sr}, i.e., $f^{\succeq_t}_{s,r} = b$. The last two observations contradict with the assumption that $\succeq_t$ is antisymmetric. To prove part (iii), suppose that $f^{\succeq_t}_{\ell,s} = a$ and $f_{j,s} = b$ where $\ell < j$, whereas a appears below b on the main diagonal. This is possible only when $a, b \in V(a) \cap V(b)$, contradicting part (ii). Similar arguments prove the claim when bundles a and b are on the same row.

Now we prove that $f^{\succeq_t}$ is strategy-proof. It suffices to consider the deviations of one negotiator. Take any $\ell, j \in [0,1]$ such that $f(\theta_1^\ell, \theta_2^j) = f_{\ell,j} = (o_X, y)$ (see figure $3-a$). Deviating from θ_1^ℓ does not benefit negotiator 1 if he deviates to θ_1^s where $s < j$ because the outcome of $f^{\succeq_t}$ will not change. However, if negotiator 1 deviates to some $s \ge j$ and get some b, we know that b is one of the logrolling bundles in B^t_{sj}. However, all of the bundles in B^t_{sj} are unacceptable for type θ_1^ℓ of negotiator 1 since $\ell < s$, and so, not preferable to (o_X, y) by the BR property.

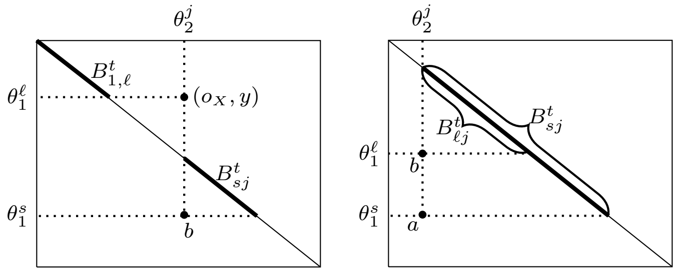

Figure 3-a Figure 3-b

Now take any $\ell, j \in [0,1]$ such that $\ell \geq j$ and $f^{\succeq_t}_{\ell,j} = b \in B^t$ (see Figure $3-b$). Deviating from θ_1^ℓ does not benefit negotiator 1 if he deviates to θ_1^s where $s < j$ because the outcome of $f^{\succeq_t}$ would be (o_X, y), which is not better than b by the BR property. If negotiator 1 deviates to some $\ell > s \geq j$ and get some a, then a must appear above b on the main diagonal (part (ii) of Lemma 4). That means $x_a\ \theta_1^\ell\ x_b$. Moreover, we must have $x_b \succeq_t x_a$because b was chosen while both a and b were available. Therefore by quid pro quo, negotiator 1 must find b at least as good as a at all admissible preferences, and thus, deviating to s is not profitable.

Finally, suppose that negotiator 1 deviates to some $s > \ell \geq j$ and get some a (see figure 3-b). Therefore, x_a beats x_b with respect to $\succeq_t$ because both a and b are in B^t_{sj} and a is chosen. Thus, a cannot be an element of $B^t_{\ell j}$ as x_b is the maximizer of $\succeq_t$ over $X_{j\ell}$. Thus, $a \in B^t_{sj} \setminus B^t_{\ell j}$, implying that a is not acceptable for type θ_1^ℓ, and so, deviating to θ_1^s is not profitable by the BR property. Hence, $f^{\succeq_t}$ is strategy-proof.

Proof of 'only if': The same arguments in the proof of Theorem 1 suffice to show that there must exist some $y \in Y$ such that $f_{\ell j} = (o_X, y)$ for all $\ell, j \in [0,1]$ with $\ell < j$. Consider now for $\ell \geq j$.

It is easy to verify that WARP, i.e., Lemma 1 in the proof of Theorem 1, also holds here. That is, if $x, x' \in A(\theta) \cap A(\theta') \neq \emptyset$, $x \neq x'$ and $x = f^X_\theta$, then $f^X_{\theta'} \neq x'$.

Lemma S.1 (Existence of t). *There exists a strictly decreasing function $t : X \to Y$ such that $f_{k,k} = (k, t(k))$ for every $k \in [0,1]$.*

Proof. We will prove that $f^Y_{\ell,\ell}\ \theta_1^Y\ f^Y_{k,k}$, i.e. $f^Y_{\ell,\ell} < f^Y_{k,k}$ for each $k \in [0,1]$ and $\ell \in [0,1]$ with $\ell > k$. If this statement is correct, then we have the desired result when we set $t(k) = f^Y_{k,k}$ for all k, including t being strict because θ_1^Y is a transitive and irreflexive relation over Y.

Suppose for a contradiction that there exists some $k \in [0,1]$ and $\ell \in [0,1]$ with $\ell > k$ such that $f^Y_{\ell,\ell} \geq f^Y_{k,k}$. First consider $f_{\ell,k}$: Efficiency and individual rationality of f and regularity of preferences imply that $f^X_{\ell,k} \in [k, \ell]$, i.e., $f^X_{k,k} \leq f^X_{\ell,k} \leq f^X_{\ell,\ell}$. Now consider $f^Y_{\ell,k}$. If $f^Y_{\ell,k} < f^Y_{\ell,\ell}$, then by monotonicity θ_2^k would deviate to θ_2^ℓ because $f_{\ell,\ell}$ gives more to negotiator 2 in both issues than $f_{\ell,k}$. Thus, we must have $f^Y_{\ell,k} \geq f^Y_{\ell,\ell}$.

If $f^Y_{\ell,k} > f^Y_{k,k}$, then by monotonicity θ_1^ℓ would deviate to θ_1^k because $f_{k,k}$ gives more to negotiator 1 in both issues than $f_{\ell,k}$. Thus, we must have $f^Y_{\ell,k} \leq f^Y_{k,k}$. But the last two inequalities imply that we have $f^Y_{\ell,\ell} \leq f^Y_{\ell,k} \leq f^Y_{k,k}$. Together with our presumption, the last inequality yields $f^Y_{\ell,\ell} = f^Y_{k,k} = f^Y_{\ell,k}$. But then because $\ell > k$, monotonicity implies that either negotiator θ_2^k profitably deviates to θ_2^ℓ to get $f_{\ell,\ell}$ or negotiator θ_1^ℓ profitably deviates to θ_1^k to get $f_{k,k}$, contradicting strategy-proofness of f. □

Therefore, a strategy-proof, efficient, and individually rational f implies a strictly decreasing function $t : X \to Y$ and a non-empty set of bundles $B^t = \{(x, t(x)) | x \in X\}$, which constitutes the set of bundles on the main (first) diagonal. For any $1 \leq j \leq \ell \leq m$ let $B^t_{j\ell} = \{(k, t(k)) \in B^t \mid j \leq k \leq \ell\}$ denote the bundles on the main diagonal between row j to ℓ. Similar to the proof of Theorem 1, we construct $\trianglerighteq$ as follows: Take any type profile $\theta = (\theta_1^\ell, \theta_2^j)$ where $1 \leq j \leq \ell \leq m$. We say $f^X_{\ell,j} \ \trianglerighteq \ x$ whenever $x \in X_{j\ell}$. WARP implies that $\trianglerighteq$ is antisymmetric and reflexive. However, it is not necessarily complete.

Similar to Lemma 3, one can verify that the binary relation $\trianglerighteq$ is transitive. Furthermore, for all $1 \leq j < \ell \leq m$, $f_{\ell,j} = \left(x^*_{X_{j\ell}}, t\left(x^*_{X_{j\ell}}\right)\right) \in B^t_{j\ell}$ where $x^*_{X_{j\ell}} = \underset{X_{j\ell}}{\boldsymbol{max}} \ \trianglerighteq$. Hence, $f = f^{\trianglerighteq}$.

By the Szpilrajn's extension theorem (Szpilrajn 1930), one can extend $\trianglerighteq$ to a complete order. This extension will clearly preserve the maximal elements in every compact subset $X_{j\ell}$ because the maximal elements in every set $X_{j\ell}$ already have a complete relation with all the elements in that set. Finally, Theorem 1 in Tian and Zhoub (1995) proves that quasi upper-semicontinuity is both necessary and sufficient for $\trianglerighteq$ to attain its maximum on all compact subsets $X_{j\ell}$, and so $\trianglerighteq$ must be quasi upper-semicontinuous.

Finally, we need to prove that $\trianglerighteq \in \Pi_\Lambda$, i.e., t and $\trianglerighteq$ are induced by the preference map Λ. To prove that $\trianglerighteq$ and t satisfy the first part of Definition 5, let $\ell, j \in [0, 1]$ be two distinct alternatives and $\ell \ \trianglerighteq \ j$. Suppose, without loss of generality, that $j \ \theta_1 \ \ell$, namely $j < \ell$. Strategy-proofness of f and consistency of preferences require that $f_{\ell,j} \ R_1 \ f_{j,j}$, or equivalently $(\ell, t(\ell)) \ R_1 \ (j, t(j))$ for all admissible $R_1 \in \Lambda(\theta_1)$ and $\theta_1 \in \Theta_1$ satisfying $j, \ell \in A(\theta_1)$, as required by part (i). To prove part (ii) suppose for a contradiction that there is some $y \in Y$ with $t(\ell) \ \theta_1^Y \ y \ \theta_1^Y \ t(j)$ such that (j, y) Pareto dominates $(\ell, t(\ell)) = f_{\ell,j}$. Because both of these bundles are acceptable at the profile $(\theta_1^\ell, \theta_2^j)$, the existence of such bundle, i.e., (j, y), contradicts with the presumption that f is efficient.

We now prove that $\trianglerighteq$ and t satisfy the second part of Definition 5. First recall that all sets of the form $X_{j\ell}$ with $1 \leq j \leq \ell \leq m$ designate all the connected subsets of X. By the construction of $\trianglerighteq$ we already know that every doubleton $\{x, x'\} \subseteq X_{j\ell}$ has a least upper bound in $X_{j\ell}$, which is $x^*_{X_{j\ell}}$, and thus the poset $(S, \trianglerighteq)$ is a semilattice for all connected subset S of X. Hence, $\trianglerighteq \in \Pi_\Lambda$. ■

Modeling Conflicting Preferences

Let X be a nonempty set of available alternatives, and Θ be the set of all linear orders on X. Define $max(\theta)$ as the maximal element of the preference ordering $\theta \in \Theta$, namely if $x^\star = max(\theta)$, then $x^\star \; \theta \; x$ for all $x \in X \setminus \{x^\star\}$. Therefore, a ***two-person, single-issue dispute*** (dispute in short) problem is a list $D = (\theta_1, \theta_2, X)$ where $\theta_i \in \Theta$ for $i = 1, 2$ and $max(\theta_1) \neq max(\theta_2)$.

For any nonempty subset $\widetilde{X} \subseteq X$, let $\theta|_{\widetilde{X}}$ denote the restriction of the preference ordering $\theta \in \Theta$ on $\widetilde{X}$. Therefore, define $\widetilde{D} = (\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{X})$ to be a dispute reduced from $D = (\theta_1, \theta_2, X)$ whenever $\widetilde{X} \subseteq X$ and $\widetilde{\theta}_i = \theta_i|_{\widetilde{X}}$ for $i = 1, 2$.

Proposition S.1. *By eliminating all the Pareto inefficient alternatives, any dispute D can be reduced to an equivalent dispute $\widetilde{D}$ where the negotiators' preferences are diametrically opposed.*

A similar result, which we omit for brevity, holds for two-person, multi-issue disputes whenever preferences over bundles satisfy monotonicity.[6]

Proof. Let $\widetilde{A} \subseteq A$ be the set of alternatives that survive the elimination of Pareto inefficient alternatives. That is, none of the alternatives in $\widetilde{A}$ is Pareto inefficient. Renumber the elements in $\widetilde{A}$, and suppose, without loss of generality, that $\widetilde{A} = \{x_1, ..., x_m\}$ where $m \geq 2$, and negotiator 1 ranks alternatives as $x_k \; \widetilde{\theta}_1 \; x_{k+1}$. If x_m is not the best alternative for $\widetilde{\theta}_2$ on $\widetilde{A}$, then there must exist some x_k where $k < m$ such that $x_k \; \widetilde{\theta}_2 \; x_m$. But this contradicts the assumption that x_m is not Pareto inefficient. Thus, negotiator 2 must rank x_m as the top alternative. With similar reasoning, if x_{m-1} is not negotiator 2's second-best alternative, then it must be Pareto inefficient, contradicting the assumption that x_{m-1} survives after the deletion of Pareto inefficient alternatives. Iterating this logic implies that the rankings of the negotiators must be diametrically opposed. □

The Revelation Principle

A mediation mechanism with veto rights $\Gamma = \big(S_1, S_2, g(.)\big)$ is a collection of action sets (S_1, S_2) and an outcome function $g : S_1 \times S_2 \to X \times Y$. The mechanism Γ combined with possible types (Θ_1, Θ_2) and preferences over bundles (R_1, R_2) with $R_i \in \Lambda(\theta_i)$ for all i defines a game of incomplete information. A strategy for negotiator i in the game of incomplete information created by a mechanism Γ is a function $s_i : \Theta_i \to S_i$.

Lemma S.2 (Revelation Principle in Dominant Strategies). *Suppose that there exists a mechanism $\Gamma = \big(S_1, S_2, g(.)\big)$ that implements the mediation rule f in dominant strategies. Then f is strategy-proof and individually rational.*

Proof. If Γ implements f in dominant strategies, then there exists a profile of strategies $s^*(.) = \big(s_1^*(.), s_2^*(.)\big)$ such that $g\big(s^*(\theta)\big) = f(\theta)$ for all $\theta \in \Theta$, and for all $i \in I$ and all $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}(\theta_{-i})\big) \; R_i \; g\big(s_i'(\theta_i'), s_{-i}(\theta_{-i})\big) \tag{1}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$ and all $s_i'(.), s_{-i}(.)$. Condition **??** must also hold for s^*, meaning that for all i and all $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\big) \; R_i \; g\big(s_i^*(\theta_i'), s_{-i}^*(\theta_{-i})\big) \tag{2}$$

[6]See Section 3 for the formal definition of monotonicity.

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$. Because $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, the last inequality implies that for all i and all $\theta_i \in \Theta_i$,

$$f(\theta_i, \theta_{-i}) \ R_i \ f(\theta_i', \theta_{-i}) \tag{3}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$.

Moreover, because mechanism Γ allows each negotiator to veto the proposed bundle and receive the outside options in each issue, there also exists a deviation strategy $\hat{s}_i(.)$ for any strategy $s_i(.)$ such that $g(\hat{s}_i(\theta_i), s_{-i}) = (o_X, o_Y)$ for all $\theta_i \in \Theta_i$ and all $s_{-i} \in S_{-i}$. The idea is that the negotiator i plays in $\hat{s}_i(.)$ exactly the same way in $s_i(.)$ (for all θ_i's) until the ratification stage and vetoes the proposed bundle.

Therefore, if $\hat{s}_i(.)$ is such a deviation strategy for $s_i^*(.)$, then condition **??** must also hold for $\hat{s}_i(.)$, implying that for all i and $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\big) \ R_i \ g\big(\hat{s}_i(\theta_i'), s_{-i}^*(\theta_{-i})\big) = \big(o_X, o_Y\big)$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$. Because $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, the last condition means that for all i and all $\theta_i \in \Theta_i$,

$$f(\theta_i, \theta_{-i}) \ R_i \ \big(o_X, o_Y\big) \tag{4}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$. Hence, conditions **??** and **??** imply that f is strategy-proof and individually rational.[7] □

Possibility without bargaining ranges

The bargaining ranges assumption provides tractability in our analysis. One might wonder what role this assumption plays in obtaining a possibility result. We provide an example to show that the bargaining ranges property is not necessary for a possibility result. (Also see Example 3 in the main text.)

Example 1 (Possibility despite the failure of bargaining ranges): Suppose that $m = 3$ and consider our model with monotonic preferences satisfying the strong form of quid pro quo assumed in Section 5. We expand this domain of preferences by adding preferences that satisfy the following rankings:

[7] If negotiators were to approve or veto each issue separately, then we would have $f(\theta_i, \theta_{-i}) \ R_i \ \big(f_X(\theta_i', \theta_{-i}), o_Y\big)$ and $f(\theta_i, \theta_{-i}) \ R_i \ \big(o_X, f_Y(\theta_i', \theta_{-i})\big)$, where $f_Z(.)$ denotes the suggested alternative by f in issue Z, together with conditions **??** and **??**.

$\theta_1^{x_1}$	$\theta_1^{x_2}$	$\theta_1^{x_3}$	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$[(x_1, y_3)]$	$[(x_2, y_2)]$	$[(x_3, y_1)]$	$[(x_1, y_3)]$	$[(x_2, y_2)]$	$[(o_X, y_1)]$
$[(o_X, y_1)]$	$[(o_X, y_1)]$	$[(x_2, y_2), (x_1, y_3), (o_X, y_1)]$	$[(x_2, y_2)]$	$[(o_X, y_1)]$	(o_X, o_Y)
$[(x_3, y_1), (o_X, y_2), (x_2, y_1)]$	(x_1, y_3)	(o_X, o_Y)	(x_3, y_1)	(x_3, y_1)	$[(x_1, y_3), (x_2, y_2)]$
(o_X, o_Y)	(x_3, y_1)		(o_X, y_3)	$[(x_1 y_3)]$	
	(o_X, y_2)		$[(o_X, y_1)]$	(o_X, o_Y)	
	(o_X, o_Y)		(o_X, o_Y)		

While the above is one specific preference profile where the bargaining ranges assumption is violated, one can include as many preference profiles of this form to our domain as long as the relative rankings of the bundles in the brackets are preserved. (The relative rankings of the bundles within the same brackets can be chosen arbitrarily.) It is easy to verify that negotiator 1-optimal rule is still strategy-proof, efficient, and individually rational.

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