Strategy-proof Multi-Issue Mediation: An Application to Online Dispute Resolution*

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Abstract

Mediation (assisted negotiation) is the preferred alternative dispute resolution approach that has given rise to a multi-billion-dollar industry worldwide. Online dispute resolution providers rely heavily on mechanized e-Negotiation systems. We develop a novel framework where two negotiators with diametrically opposed preferences seek resolution over a main issue and a supplementary issue. Negotiators are either rational, in the sense that all alternatives in the main issue are negotiable and all outcomes are acceptable, or have reference-dependent preferences, in the sense that some alternatives in the main issue are non-negotiable and some outcomes are deal-breakers (i.e., when compared to the outside option or the status quo). Types are private information and negotiators report their bargaining ranges (set of negotiable alternatives) over the main issue. The mediation process is represented by a mechanism with voluntary participation. We characterize the full class of strategy-proof, efficient, and individually rational mediation mechanisms. A necessary and sufficient condition for the existence of such protocols is the so-called quid pro quo property; a weak condition that formulates preferences for compromise solutions.

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1 Introduction

The classic 1983 bestseller book *Getting to Yes*, by Fisher and Ury identified conflict as a growth industry and the last few decades have proved them right. Judicial systems of developing and emerging economies are often challenged with a large backlog of cases, and efficiency concerns fuel implementation of reforms that focus on increased usage of alternative dispute resolution (ADR) processes.[1] Mediation is often the preferred form of ADR due to its cost-effectiveness (time-wise and financially),[2] flexibility,[3] and confidentiality. Since the 1990s, a significant number of countries have implemented both mandatory and voluntary mediation programs to improve the efficacy of their legal systems.[4]

Mediation is a consensual negotiation process in which a neutral third party (i.e., mediator) assists disputing parties to identify underlying interests, issues, and solutions, and helps them reach an agreement short of litigation. Notwithstanding the practical conveniences it affords, mediation is often considered less formal and less transparent than binding adjudication processes such as litigation and arbitration. Legal theorists argue[5] that low visibility and lack of formal rules and structure in traditional mediation reduce the rights of less powerful participants. In a seminal work, LaFree and Rack (1996) provide empirical evidence from the small claims court mediation program in Bernalillo County in Albuquerque, New Mexico, and conclude that ethnicity and gender could be more important determinants in informal mediation than they are in adjudication. In particular, they report that white males receive significantly more favorable outcomes in mediation than minority females.[6]

A structured and rigorous view of mediation is pioneered in online dispute resolution (ODR) that often rely on automation. ODR systems resolve disputes that arise both online and off-line. In a standard ODR system, parties interact through an online platform and the mediator is usually a patented software, also known as an e-Negotiation system (ENS), that follows predetermined sets of rules embodied in a mechanized algorithm. During the Internet "bubble" of

[1] See, for example, Ali (2018) for an extensive account of the recent mediation reforms.

[2] According to Hadfield (2000), it costs a minimum of $100,000 to litigate a straightforward business claim in the US, whereas a mediation session varies from few hours to a day and even the most reputable mediators charge around $10,000 - $15,000 for a day. Also, disputants do not pay any fees for experts, witnesses, document preparation, investigation, or paralegal services, which easily make the costs pile up.

[3] It is impossible to discuss a legally "irrelevant" issue in litigation/arbitration. In mediation, however, parties can discuss and negotiate issues that are not directly linked to the case.

[4] Several provinces of Canada, most notably Ontario, refer civil actions, which are subject to case management, to mandatory mediation. The mediation is conducted by a private-sector mediator and the disputants are responsible with the corresponding fees: `https://www.attorneygeneral.jus.gov.on.ca/english/courts/manmed/notice.php`. In the US, 63 federal district courts authorized the required use of mediation, out of which 12 courts mandated the use of mediation for some or all civil cases. In UK, the Small Claims Mediation Scheme is funded by HMCS (Her Majesty's Courts Service) and provides a free service for small claims cases operating in all court centers. If the parties' claim does not exceed £10,000 and agree to mediate, then a phone-based or face-to-face mediation session is arranged. In Singapore, Australia, Italy, and India court-annexed mediation takes place in the courts after parties have commenced legal proceedings, and serves as the primary method of civil dispute resolution. See Ali (2018) for an extended discussion on mandatory mediation practices in the US, UK, and aforementioned other countries.

[5] See, for example, Damaska (1975).

[6] In a similar vein, many others emphasize the factors that can cause disputant dissatisfaction that are under the direct control of mediators. As a remedy, Tyler and Huo (2002) advocate the use of fair procedures that are described as those in which decisions are viewed as *neutral, objective, and consistent*.

1999-2000, many ODR start-ups appeared and then disappeared, but since then interest in ODR has grown and its focus has expanded (Wahab et al., 2012). Over 134 ODR platforms currently operate worldwide,[7] while SmartSettle, SquareTrade, and Cybersettle are the oldest and probably the most prominent ones. It is estimated that e-commerce platforms like eBay, Paypal, Uber and Amazon resolved more than a billion disputes in 2017 through their ODR systems (Habuka and Rule, 2017). Since its founding, Cybersettle handled over 200,000 claims with a combined value in excess of $1.6B, and the City of New York uses the system since 2004 to speed their settlement process for a backlog of 40,000 personal injury claims.[8] Government use of ODR promises to be a very large market as well (Wahab et al., 2012). Government agencies, such as the National Mediation Board[9] and the Office of Government Information Services[10] in the United States, are adopting and promoting ODR as an effective method of resolving problems with citizens. In the US and Canada, 27 courts either partially or fully integrated ODR into their systems.[11] Rapid technological developments and worldwide changes brought by the Covid-19 pandemic have shown that ODR may arguably be the inevitable future of dispute resolution in the new millennium.[12]

In principle, ODR systems are ideal platforms to deliver impartial, consistent, and fair outcomes since the human factor (i.e., the mediator) is taken out of the equation and replaced with a set of reliable and objective rules and procedures. The task of automating a negotiation process is not a simple one, and this is evidenced by a myriad of systems (mostly still research efforts) around the world (Thiessen et al. 2012). However, existing systems are vulnerable to strategic manipulation, and negotiators face a daunting task of finding optimal strategies. This weakness is acknowledged by the experts in the field:

> "*A concern with the use of ENS is the possible effects of gaming and cheating. By supplying false information concerning the range of issues over which they are willing to negotiate, the results will be distorted.*" Thiessen et al. (1998).

These distortions may cause severe inefficiencies: Negotiators may fail to achieve the best possible solution despite successfully reaching a resolution. Thiessen et al. (1998), the developers of the popular SmartSettle protocols, defend the current systems in this account as follows:

> "*It is not clear from various experiments carried out that these distortions will always be to the benefit of the cheater. It may turn out, however, that if everyone cheats, the alternatives ENS generates may be negotiable and therefore useful in the negotiation process even though they may not be truly equivalent or efficient.*"

[7]See http://odr.info/provider-list/ (last visit December 11, 2022).

[8]Online Dispute Resolution Advisory Group 2015 report: https://www.judiciary.uk/wp-content/uploads/2015/02/Online-Dispute-Resolution-Final-Web-Version1.pdf (last visit December 11, 2022).

[9]See https://nmb.gov/NMB_Application/ (Last visit December 11, 2022).

[10]See https://www.archives.gov/ogis (Last visit December 11, 2022).

[11]See http://odr.info/courts-using-odr/ (last visit December 11, 2022).

[12]One of the pioneers and the first director of the ODR systems at eBay and PayPal, Colin Rule, famously wrote (see Rule (2014)) "*Now that society has embraced technology so thoroughly, the key question for dispute resolution professionals is, how can we leverage technology to best assist parties in resolving their disputes? Online Dispute Resolution is no longer a novelty—it is now arguably the future of Alternative Dispute Resolution.*"

While pioneers in the field remain optimistic, a platform that is prone to gaming may produce systematically unfair and inefficient outcomes, or may even declare an impasse for an otherwise resolvable dispute. Infrequent users (e.g., customers in e-commerce disputes, individual plaintiffs against companies and government agencies) may be disadvantaged when they face experienced users who have accumulated enough expertise about how to game the system. Although designing *fair* and *efficient* e-Negotiation systems is an active research interest in a highly interdisciplinary domain, existing models do not take incentive considerations into account. Inspired by the structured mediation programs that are offered by the ODR systems, we follow a mechanism design approach to develop a tractable framework in search for efficient, incentive compatible, and impartial mediation mechanisms.

Mechanism design has been successful in many applications, most notably in market design for auctions and matching. We adopt an *ordinal* approach whereby negotiators reveal only their bargaining ranges (i.e., range of acceptable outcomes) rather than their full-fledged (cardinal) preferences. There are three advantages of this approach. First, rather than restricting players' preferences to a specific transferable utility setting,[13] we maintain a basic common implication of any monotonic preferences in a conflict situation. Doing so allows us to characterize all classes of preferences that would support a possibility result and thereby support both transferable and nontransferable utility frameworks. Second, it is genuinely simple to implement ordinal mechanisms in practice, which is particularly important when agents are boundedly rational.[14] Third, the ordinal approach together with dominant strategy implementation (i.e., strategy-proofness) makes it possible to avoid the famous critique of Wilson (1987) by providing "detail-freeness" and "robust incentives" to participants.[15] In fact, despite the availability of well-known strategy-proof mechanisms in a number of other contexts, we are not aware of any prior work that studies strategy-proof negotiation mechanisms. (See Section 6 for a detailed discussion of the related literature.)

The backbone of our formal setting consists of two negotiators that are in a dispute over two issues, a main and a supplementary issue. According to negotiation experts, success in mediation lies in parties' ability of expanding the pie and finding integrative (win-win) outcomes, which necessitates the idea of multi-issue negotiation and logrolling (e.g., Malhotra and Bazerman, 2008; Jackson et al., 2021, and Bochet et al. 2021).[16] These two issues are represented by

[13]Much of the traditional mechanism design approach to bargaining, pioneered by the seminal work of Myerson and Satterthwaite (1983), equips negotiators with quasi-linear utility. Although money is an important issue in disputes, it is rarely the only issue (Malhotra and Bazerman, 2008). This underscores the necessity of a setting that can also admit nontransferable utility specifications and non-monetary issues.

[14]There is a large body of experimental evidence that finds that the representation of preferences by VNM utility functions may be inadequate; see, for example, Kagel and Roth (2016) for a survey. This literature argues that the formulation of rational preferences over lotteries is a complex process that most agents prefer not to engage in if they can avoid it.

[15]While stressing the powerful insights that mechanism design offers in bargaining problems, Ausubel et al. (2002) voice a similar concern: "*... Despite these virtues, mechanism design has two weaknesses. First, the mechanisms depend in complex ways on the traders' beliefs and utility functions, which are assumed to be common knowledge. Second, it allows too much commitment. In practice, bargainers use simple trading rules—such as a sequence of offers and counteroffers—that do not depend on beliefs or utility functions.*" See Carroll (2019) for an excellent survey of recent robust mechanism design techniques.

[16]See Section 5.1 for the impossibility of strategy-proof, efficient and individually rational mediation with a single issue.

discrete sets $\mathbf{X}$ and $\mathbf{Y}$. Elements of these sets are called *alternatives* and each alternative represents a settlement for that issue. Each negotiator has a commonly known ranking over the alternatives, and these rankings are diametrically opposed.[17] In an asset division problem, the main issue may represent the shares from major assets whereas the supplementary issue may represent shares from minor assets. (See Table 1 for a list of examples in practice.)

When it is common knowledge that both negotiators are rational, in the sense that they find all alternatives in each issue negotiable and all bundles acceptable, then our mediation problem boils down to a classic fair division problem. However, there is a broad consensus among both behavioral economists and decision theorists that agents behave in the same situation quite differently, depending on what *reference* they have in the form of an entitlement, endowment or default option.[18] Specifically, in the legal arena and the context of pretrial negotiations, reference-dependent behavioral explanations are widely used and often with greater predictive power than standard choice models (Robbennolt, 2014). For example, although traditional bargaining theory asserts that asymmetric information poses a major obstacle to settlements, it is widely acknowledged in law and economics that prospect theory (Kahneman and Tversky, 1979) provides a more direct and powerful explanation for the fact that great majority of the cases do in fact settle (see, e.g., Zamir and Teichman, 2014). This motivates the presence of boundedly rational negotiator types in our model along with rational types. Moreover, introducing boundedly rational types serves as a way to perturb the mediation problem and study "type-robust" mechanisms.

Following the seminal work of Masatlioglu and Ok (2014), we allow negotiators to possess reference-dependent preferences whereby bundles are evaluated from the perspective of an *outside option*, which is commonly referred as the BATNA (Best Alternative to a Negotiated Agreement) in the field.[19] A rational negotiator finds all alternatives in the main issue negotiable and all outcomes (i.e., bundles) acceptable. A boundedly rational negotiator, however, is *psychologically constrained* in that she finds some alternatives in the main issue inferior to her outside option, and declares them *non-negotiable*. Bundles including such alternatives are *unacceptable*, and would lead to the failure of mediation. A negotiator's type determines her private bargaining range. Depending on type realizations, private bargaining ranges may not overlap, and so *a zone of mutual gain* need not always exist. Therefore, the main issue has *uncertain gains from*

[17]From a technical perspective, relaxing the tension between efficiency, individual rationality and strategy-proofness by introducing an additional issue makes perfect sense as negotiators could then be asked to consider concessions in one issue for a favorable treatment (i.e., compensation) in the other, so they will be disincentivized from gaming the system. This intuition is key to the positive results in our analysis and is consistent with the theoretical (e.g., Myerson and Satterthwaite 1983, Jackson et al. 2021) and empirical literature (e.g., Backus et al. 2019, 2020) on bargaining.

[18]The empirical and experimental literature on reference-dependent individual decision making is extensive; see Camerer (1995) and Sugden (1999) for two classic surveys. In a great variety of applications (such as insurance, stock and labor markets) economic behavior is much better explained by reference dependence; see Köszegi (2014) for a recent overview.

[19]A negotiator's BATNA represents her private information and subjective expectations from alternative forms of resolution of the dispute should she walk away from mediation. According to negotiation experts success in negotiation is often viewed as being commensurate with how one can leverage her outside option. Therefore, in a negotiation framework, outside option or BATNA is a natural driving force for reference dependence. See, for example, Fisher and Ury (1981) for extensive discussions on how a negotiator should frame and bargain her position through her BATNA.

mediation. All alternatives in the supplementary issue, however, are negotiable for all types of negotiators, and so the supplementary issue has *certain gains from mediation*.

In keeping with the practice of ODR, we assume that a (mediation) mechanism maps the negotiators' private bargaining ranges in the main issue to a recommendation; either a bundle, involving an alternative for each issue, or an impasse (i.e., the outside option). A strategy-proof mechanism makes truthful declaration of one's bargaining range a dominant strategy. An efficient mechanism never recommends a bundle that can be (Pareto) improved upon. An individually rational mechanism never offers an alternative that falls outside a negotiator's declared bargaining range. Consistent with many ODR platforms, negotiators are not required to report their private preferences over the bundles. Rather, for the sake of robustness, we maintain the assumption that a negotiator's utility function can be *any* strict and monotonic function over the set of bundles. We then search for mechanisms under which, across all strict and monotonic preferences, both negotiators voluntarily participate; declare their bargaining ranges truthfully; and no negotiator can be made better off without making her counterpart worse off. If the negotiators' utility functions were public information, which is a common assumption in the mechanism design literature, then our mechanisms would be strategy-proof, ex post efficient and individually rational in the usual sense.

Our first main result is a complete characterization of the class of strategy-proof, efficient, and individually rational mechanisms (Theorem 1). These mechanisms operate through an exogenously specified *precedence order* (i.e., a partial hierarchy) over the alternatives in the main issue, and make offers from a special set of bundles, the so-called *logrolling bundles*. As the precedence order varies, the characterized class of mechanisms span what we refer to as the *family of logrolling mechanisms*. A visual characterization of this family demonstrates that a mechanism belongs to the family if and only if its matrix representation can be partitioned into rectangular regions (Theorem 3). The visual characterization simplifies the mechanics of the logrolling mechanisms and transforms them into easy-to-read diagrams that can be sequentially implemented as a menu of offers akin to those used by many online platforms such as eBay's SquareTrade. This practical simplicity can make mediation more accessible and comprehensible.

On the contrary, algorithms of current ODR platforms are mostly unknown to users (see Section 1.1 for a brief overview). Some platforms (e.g., Cybersettle) choose not to disclose their algorithms because of patent infringement concerns. Others (e.g., SmartSettle) also adopt a similar "black box" approach because their algorithms involve sophisticated and unintuitive integer optimization techniques. Designers of SmartSettle openly state that their multi-issue e-Negotiation system cannot be used to its full potential by novices without the assistance of a facilitator (Lodder and Thiessen, 2003). This, however, may pose a serious concern from an economic design perspective because it is widely acknowledged in the literature that the simplicity and the transparency of the underlying mechanics of a mechanism is crucial for its efficacy.[20]

[20] To this end, Li (2017) introduces the notion of obvious strategy-proofness which allows to distinguish even among those mechanisms that are strategy-proof. See also Pycia and Troyan (2019), Carroll (2019), and Pycia and Ünver (2020).

We then ask what is the minimum amount of information the mediator needs to have about the negotiators underlying preferences—necessary and sufficient conditions—to guarantee the existence of suitable mechanisms. Our second main result (Theorem 2) is a complete characterization of the classes of preferences that admit strategy-proof, efficient, and individually rational mechanisms. The necessary and sufficient condition is the so-called *quid pro quo* property. This property imposes a form of substitutability between the main and supplementary issues.[21] It entails that issue $\mathbf{Y}$ is rich enough so that a negotiator is able to make concessions in issue $\mathbf{X}$ to receive a more preferred alternative in issue $\mathbf{Y}$; e.g., for any pair of negotiable alternatives x and x' of $\mathbf{X}$, there exists a corresponding pair of alternatives y and y' of $\mathbf{Y}$ such that when bundled together, (x', y') is preferred over (x, y), although x is individually preferred over x'. Such reversals in the preference domain should induce a partial order and a semilattice structure on $\mathbf{X}$. An important takeaway from Theorem 2 is that not all multi-issue disputes admit good mechanisms, and in this sense, not all cases are *solvable*, despite the best efforts of the designers. It all boils down to the negotiators' underlying interests and substitutability of the issues.[22] Quid pro quo constitutes the limits of solvable disputes. If we map our discrete model into a classic exchange economy, where alternatives represent quantities of goods, then well-known utility functions, such as a CES or a quasi-linear utility, satisfy quid pro quo (Examples 1 and 3).

The family of logrolling mechanisms nests interesting special members. When the precedence order is in line with the preference of a given negotiator over the logrolling bundles, we obtain the corresponding *negotiator-optimal mechanism*. A negotiator-optimal mechanism represents situations when a mediator may be categorically biased toward one party in the dispute. Hence, we introduce a fairness criterion that is useful in judging the impartiality of the mediation processes. A central member of the family of strategy-proof, efficient, and individually rational mediation mechanisms that satisfies this criterion (Theorem 4) is the so-called *constrained shortlisting* mechanism. This mechanism recommends the *median logrolling bundle* when it is mutually negotiable, and when it is not, favors the least-accepting negotiator.

1.1 Online Dispute Resolution and e-Negotiation Systems

Before presenting our model and theoretical results, we provide a brief overview of the fundamental aspects of e-Negotiation protocols. The essential commonalities of these protocols motivate some of our modeling choices. ODR protocols usually take issues and possible alternatives in every single issue as given. This information is often solicited when parties describe the dispute at the time they first request the service of the ODR platform. The negotiation problem is then created by populating this information on the system. Aside from this preliminary step, protocols generally involve three common steps: (1) *elicitation*, (2) *proposal*, and (3) *ratification*.

[21] The property can be viewed as the nontransferable utility analogue of the *possibility of compensation* assumption in a transferable utility model, see, for example, Thomson (2016).

[22] Consider, for example, a scenario where alternatives of issue $\mathbf{Y}$ have little appeal for the negotiators compared to those in issue $\mathbf{X}$ (e.g., preferences are lexicographic over the two issues). Then there is little reason to suspect that the impossibility in the single-issue case will be overturned in the two-issue world. In fact, quid pro quo property fails to hold in such preference domains.

The process ends either when parties unanimously accept a proposal, or if no mutual agreement is reached after several iterations of an "*elicitation-proposal-ratification*" cycle.

The goal of the elicitation step is to obtain parties' private information. First, parties are asked about their bargaining ranges for each issue, with the understanding that the mediator's proposal will never include an alternative outside this range. Negotiators' bargaining ranges are elicited by asking each negotiator to choose an alternative that is the least acceptable for her. Given the negotiators' positions (i.e., how negotiators rank the alternatives) the system infers the set of negotiable alternatives once they declare their bargaining ranges. ODR platforms commonly make the implicit assumption that negotiators' ordinal rankings over alternatives are monotonic in the sense that an alternative that delivers more is always better. This automatically implies that negotiators' rankings of the available alternatives in a given issue are diametrically opposed. Depending on the ODR platform, parties may be further asked to report their preferences over issues to indicate how they trade off one issue against another. SmartSettle, for example, elicits cardinal preferences in the form of utility (satisfaction) points over issues and alternatives. Namely, each negotiator is asked to "bid" a point value (between 0 and 100) for each alternative in each issue.

Protocols usually differ in how they process all this information to make recommendations, and majority of the existing e-Negotiation protocols use a combination of cooperative game-theoretic and optimization techniques. In all existing protocols, a common theme is that any aspect of a user's input (e.g., bargaining ranges) may be modified at any time during the negotiation before a proposal is accepted by both negotiators. We discuss three representative examples from the field.

The protocols used by the popular SmartSettle system are based on optimization algorithms that use mixed-integer programming techniques.[23] The system categorizes solution packages (bundles of alternatives) non-negotiable if they include alternatives that are outside of the declared bargaining ranges or utility points fall short of the minimum scores privately declared by the disputants. The system never recommends these bundles.

A second well-known example is the Adjusted Winner by Brams and Taylor (1996) which is based on cooperative game-theory. It was later adopted by Bellucci and Zeleznikow (2005) to model Australian family law based on the repository of cases in the Australian Institute of Family Studies and subsequently applied to international disputes, enterprise bargaining, and company mergers. The algorithm is a point allocation procedure that aims to distribute items to the negotiators on the premise of who values the issue the most. At the outset the negotiators are required to distribute 100 points across the range of issues. The algorithm first identifies the issue that the disputants are furthest apart and allocates the item in this issue to the party who values it the most. Then it finds the next issue where the disputants are furthest apart and allocates the item in that issue to the party who values it the most, and so on.[24]

[23]Although the complete details of the algorithm is not publicly available, a subset of the linear equations are provided in Thiessen and Loucks (1992).

[24]While the Adjusted Winner is a manipulable cardinal scheme, the optimal responses of the players to the choices of an opponent are close to being truthful. It also provides a guarantee of each player's obtaining at least 50 of 100 points, even if its opponent knows exactly its point allocation and responds optimally to maximize its

A third representative example is SquareTrade, a platform that has been eBay's contractor on dispute resolution and the leading ODR provider for consumer mediation since 1999.[25] The main difference of the SquareTrade from the previous two examples is that it does not involve any preference elicitation. Specifically, the SquareTrade dispute resolution process consists of two stages. In the first stage, it presents the claimant a list of possible solutions (alternatives or bundles) and asks her to select the ones that she finds negotiable. Upon agreeing to participate in the process, the other party is then asked to do the same. If at least one solution is mutually acceptable, then the process ends. Incidentally, this process is a simpler and less refined version of the central mechanism (i.e., a logrolling mechanism) we propose and characterize in this paper.[26] If parties cannot reach an agreement in the first stage, then the protocol allows parties to exchange visible optimistic proposals, defining the bargaining range in the second stage. The system then generates suggestions that fall into the bargaining range. Parties may continue to exchange visible proposals or contribute their own suggestions to the mixture of standing proposals. The process terminates when all parties accept at least one standing proposal.

A strand of market design literature that can offer valuable insights in the context of ODR includes the recent works on *multi-item assignment*, such as course allocation at business schools. A common allocation method in practice is a *course-bidding mechanism* where students are asked to allocate an artificial currency endowment across different courses, and courses are assigned to highest bidders. Both theoretical and experimental research have shown that such auction mechanisms can perform rather poorly due to the perverse incentives they generate.[27] A major insight from that context immediately carries over to dispute resolution: When a bidding mechanism is used, a disputant can find it strategically advantageous not to "waste" points on less-contested issues despite having a truly high valuation of these issues. This in turn translates into strategic reports that are not representative of true preferences. Such incentive shortcomings of point-based course allocation systems played role in the recent replacement of the course-bidding mechanism at the Wharton School (University of Pennsylvania) with the Approximate Competitive Equilibrium from Equal Incomes (A-CEEI) mechanism of Budish (2011), which has superior incentive properties.

own share. See Chapter 4 in Brams and Taylor (1996).

[25]SquareTrade has resolved millions of disputes across 120 countries in 5 different languages (Cortes 2014).

[26]See Section 3.4 for a detailed discussion for practical relevance.

[27]See, for example, Sönmez and Ünver (2010) and Krishna and Ünver (2008).

2 Multi-Issue Mediation with Reference-Dependent Types

The first part of this section presents the model and the terminology adopted throughout the paper. We defer our interpretation and discussion of key assumptions to the second part.

2.1 The Framework

Issues and alternatives: Two negotiators are in a dispute over two issues. Let $\mathbf{I} = \{1, 2\}$ be the set of negotiators. We refer to negotiators as "she" and to the mediator as "he". We denote a generic negotiator by $i \in \mathbf{I}$ and her opponent by $-i \in \mathbf{I} \setminus \{i\}$. Let $\mathbf{X} = \{x_1, ..., x_m\}$ and $\mathbf{Y} = \{y_1, ..., y_n\}$ denote the finite set of ordered **alternatives** in the **main** and the **supplementary** issues, respectively, where $n \geq m \geq 2$. The supplementary issue becomes obsolete and irrelevant when the negotiators cannot agree on the main issue.[28] Depending on the type of negotiation, alternatives may represent prices, quantities, quality levels, shares of assets, possible delivery dates, employment positions, or various other contractual terms (see Table 1 below for more specific examples).

It is public information that alternative x_k (respectively, y_k) for $k \in \mathbb{N}$ is the kth best alternative for Negotiator 1 in the main issue (respectively, in the supplementary issue), and Negotiator 2 has diametrically opposed preferences in each individual issue:[29] Namely, x_m (respectively, y_n) is Negotiator 2's best alternative, x_{m-1} (respectively, y_{n-1}) is her second-best alternative in the main issue (respectively, supplementary issue), and so on. We use x, x' and y, y' to denote generic alternatives in issues $\mathbf{X}$ and $\mathbf{Y}$, respectively, whenever it is not necessary to specify the rank of these alternatives.

Boundedly rational negotiator types: We assume two types of negotiators depending on how one views the alternatives in the main issue in reference to the status quo outcome of disagreement. All alternatives are negotiable for a *rational negotiator type*. However, there is a positive (and potentially very small) probability that each negotiator is of a *boundedly rational type* (or reference dependent/status quo biased) for whom some alternatives in the main issue are not up for negotiation. Following Fischer and Ury (1983), we let the auxiliary *outcome* ϕ represent a negotiator's Best Alternative to A Negotiated Agreement (BATNA), which we refer as the **outside option** using the convention in economics. An alternative in the main issue that is inferior to ϕ is a *non-negotiable alternative*. An alternative is a *negotiable alternative* if it is not one of the non-negotiable alternatives. For a given negotiator, the set of alternatives that she finds negotiable determines her type. Let x_k^i where $k \in \mathcal{I} = \{1, .\ \ldots, m\}$, denote the type of negotiator i whose least negotiable alternative is x_k. That is, Negotiator 1 with type x_k^1 ranks ϕ immediately above alternatives x_{k+1} through x_m, and Negotiator 2 with type x_k^2 ranks ϕ immediately above alternatives x_{k-1} through x_1. Let $\mathbf{T}_i$ be the set of all types of negotiator i.

[28] Alternatively, issues of the dispute may be independent from one another, and so negotiators may mutually agree on an alternative in one issue although they cannot agree on an alternative in the other issue. All our results would carry over to such a framework.

[29] This is a mere implication of monotonicity. For example, in asset division, each negotiator would prefer to have larger shares; in settlement/price bargaining, one side would prefer higher prices as opposed to the other; in commerce/construction, the service provider would prefer later delivery dates contrary to the receiver; etc.

Type x^1_m (respectively, type x^2_1) is the rational type of Negotiator 1 (respectively, of Negotiator 2) since she finds all alternatives negotiable. Each negotiator privately knows her true type, and we assume, without loss of generality, that all types find at least one alternative in the main issue negotiable.

More formally, let $\mathbf{N}(x^1_k) = \{x_\ell \in \mathbf{X} |\ \ell \leq k\}$ and $\mathbf{N}(x^2_k) = \{x_\ell \in \mathbf{X} |\ \ell \geq k\}$ denote the set of all **negotiable alternatives** for type x^1_k of Negotiator 1 and type x^2_k of Negotiator 2, respectively. Therefore, $\mathbf{X} \setminus \mathbf{N}(x^i_k)$ denotes the set of all **non-negotiable alternatives** for type x^i_k of negotiator i. We use $t_i \in \mathbf{T}_i$ for a generic type of negotiator i. Let $\mathbf{T} = \mathbf{T}_1 \times \mathbf{T}_2$ denote the set of all type profiles. All alternatives in the supplementary issue are negotiable.[30]

Reference-dependent preferences: The set of all **outcomes** of the mediation, denoted by $\mathbf{B}$, consists of the set of all **bundles** and the outside option ϕ. Namely, $\mathbf{B} = (\mathbf{X} \times \mathbf{Y}) \cup \{\phi\}$. **Acceptable bundles** of type t_i of Negotiator i is denoted by $A(t_i) = \{(x, y) \in \mathbf{X} \times \mathbf{Y} |\ x \in \mathbf{N}(t_i)\}$. Let $\mathcal{U}_1$ (respectively, $\mathcal{U}_2$) denote the set of all *strict* and *decreasing* (respectively, *increasing*) functions from $\mathbf{B}$ into $\mathbb{R}$.[31] Negotiator i's preferences over outcomes are represented by some utility function $u_i \in \mathcal{U}_i$ regardless of her type. Namely, preferences of the rational and boundedly rational types coincide over all outcomes with the exception of the outside option. Negotiator i privately knows her utility function and believes that her opponent's utility is one of those in $\mathcal{U}_{-i}$. We denote the set of all **admissible utility profiles** by $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$.

To describe the behavior of boundedly rational negotiator types, we follow Masatlioglu and Ok (2014) who extend the rational choice model to incorporate the widely documented status quo bias phenomenon. Let $C_i : \mathcal{U}_i \times \mathbf{T}_i \times 2^{\mathbf{B}} \setminus \{\varnothing\} \to \mathbf{B}$, and so $C_i(., u_i, t_i)$ denote the choice behavior of type t_i of Negotiator i when her utility function is u_i. Therefore, type t_i of Negotiator i is **boundedly rational** in the sense that for all $u_i \in \mathcal{U}_i$ and all $S \in \mathbf{B} \setminus \{\varnothing\}$,

$$C(S, u_i, t_i) = \arg\max u_i(b) \quad \text{subject to} \quad b \in S \cap A(t_i)$$

whenever $S \cap A(t_i) \neq \varnothing$, and

$$C(S, u_i, t_i) = \phi,$$

otherwise.

Namely, the boundedly rational types are (psychologically) constrained utility maximizers over the set of bundles that are "appealing" from the perspective of the status quo outcome ϕ. Note that the rational negotiator finds all bundles acceptable, and so, she never chooses the outside option, unless it is the only option available in S.

Standard definitions for strategy-proofness, efficiency and individual rationality are based on agents' utility functions, assuming that utilities perfectly reflect their choice behaviors. However,

[30] The supplementary issue represents a dimension of the negotiation that is of secondary importance (relative to the main issue) such that no alternative in this issue would lead to a breakdown of the negotiation. Alternatives in the supplementary issue can however be leveraged for compensation when bundled together with alternatives in the main issue.

[31] A function $u : \mathbf{B} \to \mathbb{R}$ is **strict** if for any $b, b' \in \mathbf{B}$, $u(b) = u(b')$ if and only if $b = b'$. It is **increasing** (respectively, **decreasing**) if for all $x_k, x_{k'} \in \mathbf{X}$ and all $y_\ell, y_{\ell'} \in \mathbf{Y}$, where $k \geq k'$ and $\ell \geq \ell'$, we have $u(x_k, y_\ell) \geq u(x_{k'}, y_{\ell'})$ (respectively, $u(x_k, y_\ell) \leq u(x_{k'}, y_{\ell'})$).

this does not hold in our setup. Namely, choice behaviors of two different types of the same negotiator may be different, as they face different behavioral constraints, although their utility functions are the same. A resolution for this is to define these key concepts through preferences that *rationalize* the choice behavior of each type of each negotiator.[32] We formalize these ideas next.

For any $u_i \in \mathcal{U}_i$, define a utility function $u_i^R : \mathbf{B} \times T_i \to \mathbb{R}$ as follows: For all $t_i \in T_i$, $b \in A(t_i)$, and $b' \in \mathbf{B} \setminus A(t_i)$, we have $u_i^R(b,t_i) = u_i(b)$ and $u_i^R(b,t_i) > u_i^R(\phi,t_i) \geq u_i^R(b',t_i)$. The last inequality, which we refer to as **deal-breaker property**, implies that all acceptable bundles are preferable to the outside option, and the outside option is preferable to all remaining bundles.

Remark 1. *Choice behavior of every type t_i of Negotiator $i \in \mathbf{I}$ is **rationalizable** by some utility function in $\mathcal{U}_i^R = \{u_i^R |\ u_i \in \mathcal{U}_i\}$. Namely, for every $u_i \in \mathcal{U}_i$, there exists $u_i^R \in \mathcal{U}_i^R$ such that $C(S, u_i(.), t_i) = C(S, u_i^R(., t_i), t_i)$ for all $S \in \mathbf{B} \setminus \{\varnothing\}$ and $t_i \in \mathbf{T}_i$.*

Proof of this remark immediately follows from the definitions. Given that $\mathcal{U}_i$ denotes the set of all admissible utility functions for Negotiator i, we denote the set of all rationalizable utility functions for i by $\mathcal{U}_i^R$. We denote the set of all **rationalizable utility profiles** by $\mathcal{U}^R = \mathcal{U}_1^R \times \mathcal{U}_2^R$.

Although $\mathcal{U}$ is a restricted subset of the set of all preference profiles, it is a very large domain of preferences. However, as commonly observed in the literature, strategy-proofness, efficiency and individual rationality may be mutually compatible over some strict subset of $\mathcal{U}$ (e.g., both negotiators have quasi-linear or Cobb-Douglass utility functions; see Examples 1 and 3). For this reason, we define these properties over a non-empty subset $\mathcal{D}$ of $\mathcal{U}$, which we call the set of **discernible utility profiles**. Our interpretation of $\mathcal{D}$ is that it is common knowledge among all three agents that the negotiators' preferences are one of those in $\mathcal{D}$. The set $\mathcal{D}^R \subseteq \mathcal{U}^R$ denotes the set of all rationalizable and discernible utility profiles. Namely, $\mathcal{D}^R = \left\{(u_1^R, u_2^R) \in \mathcal{U}^R | (u_1, u_2) \in \mathcal{D}\right\}$. We let $\mathcal{D}_i^R$ (respectively, $\mathcal{D}_i$) to be the projection of $\mathcal{D}^R$ (respectively, $\mathcal{D}$) over $\mathcal{U}_i^R$ (respectively, $\mathcal{U}_i$).[33]

Mechanisms: A direct (mediation) mechanism $f : \mathbf{T} \to \mathbf{B}$ requires each negotiator to report her type (i.e., her negotiable alternatives), and either proposes a bundle (x, y), specifying an alternative for each issue, or declares an impasse (i.e., ϕ). For convenience, a mediation mechanism f is equivalently represented by an $m \times m$ matrix $f = [f_{k,\ell}]_{(k,\ell)\in\mathcal{I}^2}$ where $f_{k,\ell} \in \mathbf{B}$. The rows (respectively, columns) of this matrix correspond to the types of Negotiator 1 (respectively, Negotiator 2). In particular, row k denotes type x_k^1 of Negotiator 1 and column k refers to type x_k^2 of Negotiator 2. Consider any $t = (x_k^1, x_\ell^2) \in \mathbf{T}$: Whenever $f(t) \neq \phi$, we use $f_{k,\ell}^{\mathbf{X}}$ and $f_{k,\ell}^{\mathbf{Y}}$ to denote the alternative that the mediation mechanism f chooses from issue $\mathbf{X}$ and $\mathbf{Y}$, respectively. Namely, $f(t) = f_{k,\ell} = \left(f_{k,\ell}^{\mathbf{X}}, f_{k,\ell}^{\mathbf{Y}}\right)$.

[32] Alternatively, these notions can be equivalently defined through agents' choice functions.

[33] More formally, $\mathcal{D}_i^R = \{U_i \in \mathcal{U}_i^R | (U_i, U_{-i}) \in \mathcal{D}^R$ for some $U_{-i} \in \mathcal{U}_{-i}^R\}$ and $\mathcal{D}_i = \{u_i \in \mathcal{U}_i | (u_i, u_{-i}) \in \mathcal{D}$ for some $u_{-i} \in \mathcal{U}_{-i}\}$.

Fix a discernible utility profile $\mathcal{D}$. Mechanism f is **strategy-proof** over $\mathcal{D}$ if for all $i \in \mathbf{I}$, $t_i \in \mathbf{T}_i$, and all $U_i \in \mathcal{D}_i^R$,

$$U_i\big(f(t_i, t_{-i}), t_i\big) \geq U_i\big(f(t_i', t_{-i}), t_i\big)$$

for all $t_i' \in \mathbf{T}_i$ and all $t_{-i} \in \mathbf{T}_{-i}$. Mechanism f is (ex post) **individually rational** over $\mathcal{D}$ if for all $i \in \mathbf{I}$, $t \in T$, and all $U_i \in \mathcal{D}_i^R$,

$$U_i\big(f(t), t_i\big) \geq U_i\big(\phi, t_i\big).$$

Mechanism f is (ex post) **efficient** over $\mathcal{D}$ if there exists no $t \in \mathbf{T}$ and no $b \in \mathbf{B}$ such that $U_i(b, t_i) \geq U_i(f(t), t_i)$ for all $i \in \mathbf{I}$ and all $U_i \in \mathcal{D}_i^R$, and the inequality is strict for at least one i and U_i.

We next introduce a series of definitions that will be key in the presentation of our main results. An injective function $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ is **decreasing** if for all $x_k, x_{k'} \in \mathbf{X}$ with $k > k'$, there are $y_\ell, y_{\ell'} \in \mathbf{Y}$ with $\ell < \ell'$ such that $y_\ell = \mathbf{y}(x_k)$ and $y_{\ell'} = \mathbf{y}(x_{k'})$.

For any non-empty subset $\mathbf{S}$ of $\mathbf{X}$ and a partial order $\trianglerighteq$ on $\mathbf{X}$, let $\underset{\mathbf{S}}{\boldsymbol{max}} \trianglerighteq$ denote the maximal element in $\mathbf{S}$ with respect to $\trianglerighteq$.[34] Namely, if $x^*_{\mathbf{S}} = \underset{\mathbf{S}}{\boldsymbol{max}} \trianglerighteq$, then $x^*_{\mathbf{S}} \trianglerighteq x$ for all $x \in \mathbf{S}$. Note that such a maximal element is not guaranteed to exist under an arbitrary (e.g., incomplete) partial order.

Finally, the tuple $(\mathbf{X}, \trianglerighteq)$ is called a **poset** (short for partially ordered set) if $\trianglerighteq$ is a partial order $\mathbf{X}$. For a poset $(\mathbf{X}, \trianglerighteq)$, we say that an element $x \in \mathbf{X}$ is an **upper bound** of a subset $\mathbf{S} \subseteq \mathbf{X}$ when $x \trianglerighteq x'$ for all $x' \in \mathbf{S}$. The **least upper bound** of $\mathbf{S}$ is the upper bound of $\mathbf{S}$ that is less than or equal to every upper bound of $\mathbf{S}$. Namely, x is a least upper bound of $\mathbf{S}$ if $x' \trianglerighteq x$ for all upper bounds x' of $\mathbf{S}$. Given a doubleton $\{x, x'\} \subseteq \mathbf{X}$, let the join of x and x', denoted by $x \vee x'$, be the least upper bound of the doubleton. A poset $(\mathbf{X}, \trianglerighteq)$ is called a **join semilattice** if every doubleton $\{x, x'\} \subseteq \mathbf{X}$ has a least upper bound in $\mathbf{X}$.

2.2 Discussion

Table 1 provides a list of common disputes and negotiations in practice to help contextualize the model. Example 1 below formalizes the first example in this list.

We model mediation for disputes with two issues and finite sets of alternatives, but our results easily extend to cases with more than two issues (see Section 5.4) or continuum of alternatives. The assumption that preferences over alternatives in each individual issue are diametrically opposed is without loss of generality. Under the requirement of efficiency, any dispute where preferences over alternatives are not diametrically opposed can be equivalently represented by a "reduced dispute" where "reduced preferences" are diametrically opposed (see Section 5.2).

The outside option (or BATNA as referred in the negotiation literature) plays a crucial role in our modeling of types and behavior, and represents various sources of asymmetric information. Whereas it can be a trader's private cost in a classical bilateral bargaining context, in the law and economics literature it often represents a negotiator's subjective expected utility from the

[34] A binary relation $\trianglerighteq$ on set $\mathbf{X}$ is called a *partial order* on $\mathbf{X}$ if $\trianglerighteq$ is transitive, reflexive, and antisymmetric.

Table 1: Some common mediation cases and relevant issues

Type of Mediation	Main Issue (**X**)	Supplementary Issue (**Y**)
Asset division (e.g., family, partnership & inheritance)	Major assets[a]	Minor assets[b]
E-commerce	Price	Delivery date and return policy
Environmental	Pollution reduction terms	Penalty or tax exemptions
Employment	Job essentials (e.g., title, location & pay)	Other terms (e.g., office, bonus, start date & leaves)
Construction	Completion requirements (e.g., date & price estimate)	Other terms (e.g., design specifications & contingencies)
Business-to-Business	Market shares (e.g., territorial division)	Business support (e.g., hardware & pricing flexibility)

Notes:
[a,b] Depending on the dispute, an asset can be *tangible* (e.g., lands, plants, equipment, buildings, cash, inventory, or stocks) or *intangible* (e.g., goodwill, patents, brand, copyrights, trademarks, licenses, or permits). Whether an asset is a major or minor asset can be based on its relative market value. Claims over a tangible asset can be measured in terms of quantities (e.g., see Example 1), shares (e.g., percentages), or liquidity equivalents.

status quo outcome (e.g., an alternative resolution to the dispute such as litigation). As such, in our modeling a negotiator's type embodies her prior claims and other extraneous private information that is not public information.

Throughout the paper we assume that boundedly rational negotiator types evaluate alternatives from the perspective of their outside option. Any alternative that is deemed inferior to the status quo is non-negotiable, and so, a bundle with a non-negotiable alternative is a deal-breaker (i.e., worse than the outside option). To understand how boundedly rational negotiators act in a mediation problem, it is useful to see how they would behave in a single-person choice problem. For this reason, consider a scenario where a negotiator must choose an alternative (i.e., a bundle) from a feasible set **B**. In Figure 1, the choice set is some arbitrary subset of $\mathbb{R}^2$, for the sake of argument, where the first coordinate represents the main issue and the second coordinate represents the supplementary issue. A rational negotiator chooses from the feasible set **B** by maximizing a utility function on **B**. (In Figure 1, a rational negotiator with continuous and quasiconcave preferences would choose b from **B**.) By contrast, our boundedly rational negotiator, whose outside option is denoted by ϕ, chooses from **B** by maximizing a utility function on **B** subject to a (psychological) constraint induced by ϕ. All outcomes to the left and below the red lines that pass through ϕ are deal-breakers for the negotiator, and so only the bundles to the north east of ϕ in Figure 1 are acceptable. (In Figure 1, for instance, the boundedly rational negotiator would declare b' as the choice from **B**.) In this particular example, we generate the different private types of the same negotiator by varying the location of ϕ over the horizontal line PQ because the ranking (i.e., the location) of the outside option with respect to the alternatives in the main issue is the negotiator's private information. In the absence of outside option, our negotiator types are rational in the standard sense.

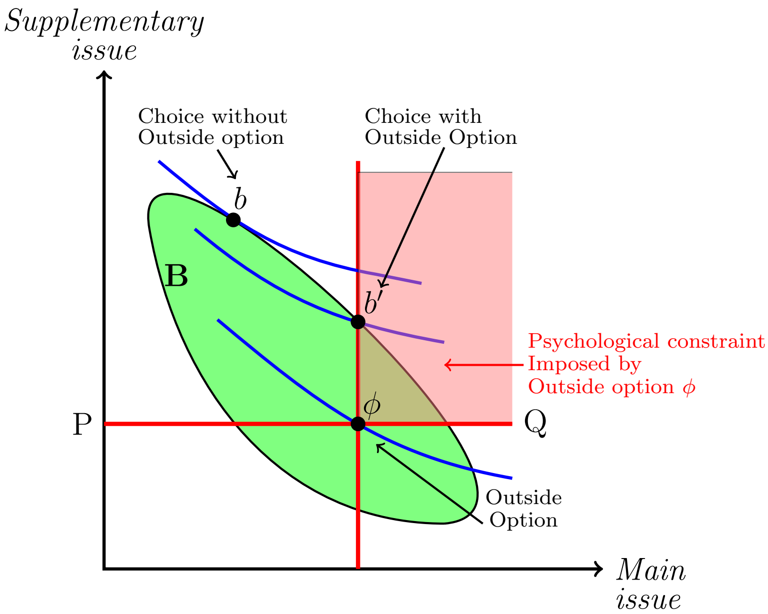


Figure 1

Mediation practices are consensual processes. Namely, negotiators have the right to opt out of mediation at any point of the process, and choose the status quo (i.e., seek alternative ways to resolve their disputes.) As a result, introducing boundedly rational types into the model of mediation is important for two reasons. First, given the plethora of empirical and experimental evidence regarding the behavioral implications of status quo bias and its wide usage in pretrial bargaining (see, for example, Masatlioglu and Ok (2014) and Zamir and Teichman (2014) and the references therein), rationality of *all* negotiators is an impractical assumption. We relax this assumption by introducing boundedly rational types. Second, the mechanisms we characterize in this paper are strategically robust to manipulation in case negotiators are actually boundedly rational. Hence, considering a richer set of behavioral types makes our analysis more appealing and relevant for practical purposes.

The outside option effectively determines a negotiator's bargaining range in the sense that any inferior alternative is a dealbreaker. This formulation is akin to *unacceptability* in compatibility-based preference settings in assignment problems, such as the dichotomous preferences in Bogomolnaia and Moulin (2004) and Roth, Sönmez, and Ünver (2006). Our preference formulation, however, is considerably less restrictive than dichotomous preferences because negotiators have freedom to strictly prefer one negotiable alternative over another. It is worth noting that dealbreaker outcomes may also arise naturally (without a reference to a status quo) in a number of negotiation problems: In many business-to-business disputes, prior commitments, reputational concerns, or industrial practices render certain options and issues non-negotiable. Similarly, in many international, interracial or inter-religious conflicts, some issues/solutions may be viewed as dealbreakers due to existing norms and values.

Monotonicity of preferences is a standard requirement and simply demands that a bundle with better alternatives in both issues is always more preferred. The assumption on strict utility functions is without loss of generality. It helps us eliminate indifferences, and so many redundant mediation rules that may arise in our characterization results with no extra insight. We assume that the negotiators' preferences over the alternatives (except the outside option) in each issue are diametrically opposed. (One may think that the alternatives in each issue are efficient divisions of a fixed surplus of known size; see the next example for further detail). Given this assumption, we can place the choice set $\mathbf{B}$ and both negotiators' indifference curves on the same graph, very much in the spirit of Edgeworth Box in a two-person exchange economy.

Mediation would potentially be a complicated multistage game between the negotiators and the mediator. The mediation protocol, whatever the details may be, produces proposals for agreement that are always subject to a unanimous approval by the negotiators. That is, before finalizing the protocol, each negotiator has the right to veto the proposal and exercise her outside option. A version of the revelation principle guarantees that we can stipulate the following type of a two-stage *direct mechanism with veto rights* without loss of generality when representing voluntary mediation.

The two stages of the protocol are called the *announcement* stage and the *ratification* stage. The announcement stage is characterized by a mechanism $f : \mathbf{T} \to \mathbf{B}$. After being informed of her type, each negotiator i privately reports her type t_i to the mediator who then proposes a bundle $f(t_1, t_2) \in \mathbf{B}$. In the ratification stage, each party simultaneously and independently decides whether to accept or veto the proposed bundle. If both negotiators accept the proposed bundle, then it becomes the final outcome. If either or both negotiators veto the proposal, then mediation fails and each party gets the outside option (i.e., ϕ). We seek direct mechanisms with veto rights in which truthful reporting of types at the announcement stage is a *dominant strategy equilibrium* and the mediator's proposals are never vetoed in equilibrium. It immediately follows from the definitions that such an equilibrium exists if and only if mechanism f is strategy-proof and individually rational.

Example 1: To illustrate the compatibility of our setup with standard utility functions, we provide an example of a dispute in which issue $\mathbf{X}$ represents a major asset and issue $\mathbf{Y}$ represents a minor asset. We later build on this example to illustrate our main results. Suppose that the negotiators need to divide positive surpluses $\bar{x}$ and $\bar{y}$ in issues $\mathbf{X}$ and $\mathbf{Y}$, respectively. Therefore, we let $\mathbf{X} = \{x_1, \ldots, x_m\}$ and $\mathbf{Y} = \{y_1, \ldots, y_n\}$ be two subsets of the real numbers, where $0 < x_m < \ldots < x_1 < \bar{x}$ and $0 < y_n < \ldots < y_1 < \bar{y}$. The following two utility specifications $u_i : \mathbf{B} \to \mathbb{R}_+$ and $v_i : \mathbf{B} \to \mathbb{R}_+$ satisfy all the key properties we imposed earlier:

- *Quasi-linear:*

$$u_1(x, y) = x^{\alpha_1} + y, \tag{1}$$
$$u_2(x, y) = (\bar{x} - x)^{\alpha_2} + (\bar{y} - y), \tag{2}$$

  where $\alpha_i \in (0, 1)$ for $i = 1, 2$, and

- *Cobb-Douglas:*

$$v_1(x, y) = x^{\gamma_1} y^{\beta_1}, \tag{3}$$
$$v_2(x, y) = (\bar{x} - x)^{\gamma_2} (\bar{y} - y)^{\beta_2}, \tag{4}$$

where $\gamma_i, \beta_i \in \mathbb{R}_{++}$ for $i = 1, 2$.

In this dispute, each negotiator prefers a larger share of any asset than her opponent (i.e., preferences are diametrically opposed in both issues). The utility values for the outside option are normalized so that $u_i(\phi) = v_i(\phi) = 0$ for all $i \in \mathbf{I}$. Functions u_1 and u_2 are standard quasi-linear utility functions. Similarly, v_1 and v_2 are classic Cobb-Douglas utility functions. Given the utility functions u_i and v_i, it is straightforward to define rationalizable utility functions u_i^R and v_i^R. For instance, for $i = 1, 2$ we set $u_i^R(\phi) = v_i^R(\phi) = 0$, and for all $(x, y) \in \mathbf{X} \times \mathbf{Y}$, $u_i^R(x, y, t_i) = u_i(x, y)\mathbf{1}_{t_i}(x)$ and $v_i^R(x, y, t_i) = v_i(x, y)\mathbf{1}_{t_i}(x)$, where

$$\mathbf{1}_{t_i}(x) = \begin{cases} 0 & \text{if } x \in \mathbf{X} \setminus \mathbf{N}(t_i) \\ 1 & \text{otherwise.} \end{cases}$$

3 Main Results

3.1 Strategy-Proof Mediation

We start with the characterization of the necessary conditions for a strategy-proof, efficient, and individually rational mechanism. Given a type profile $(x_\ell^1, x_j^2) \in \mathbf{T}$ with $j \leq \ell$, both negotiators have something to gain from mediation relative to an impasse. Therefore, we call the non-empty subset $\{x_k \in \mathbf{X} | j \leq k \leq \ell\} \equiv [x_j, x_\ell]$ of $\mathbf{X}$ as the **zone of mutual gain**.

Theorem 1. *Suppose that f is a strategy-proof, efficient, and individually rational mechanism over a non-empty set $\mathcal{D} \subseteq \mathcal{U}$. Then there exists an injective and decreasing function $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$, and a partial precedence order $\trianglerighteq$ on $\mathbf{X}$ such that for any $t = (x_\ell^1, x_j^2) \in \mathbf{T}$ where $j, \ell \in \mathcal{I}$, if $j > \ell$, then the mechanism f recommends the outside option ϕ, and otherwise (i.e., if $j \leq \ell$) the mechanism f recommends an alternative from the zone of mutual gain $\{x_j, \ldots, x_\ell\}$ that has the highest precedence according to $\trianglerighteq$, denoted by $x^*_{[x_j, x_\ell]}$, and the alternative from the supplementary issue that corresponds to $\mathbf{y}(x^*_{[x_j, x_\ell]})$. More formally,*

$$f(t) = \begin{cases} \left(x^*_{[x_j, x_\ell]}, \mathbf{y}\left(x^*_{[x_j, x_\ell]}\right)\right) & \text{if } j \leq \ell \\ \phi & \text{otherwise;} \end{cases}$$

where $x^*_{[x_j, x_\ell]} = \max\limits_{[x_j, x_\ell]} \trianglerighteq$.

Theorem 1 states that when negotiators have a non-empty mutual zone of gain in the main issue, a desired mechanism first maximizes an exogenous partial order $\trianglerighteq$ to determine the chosen alternative for the main issue, and then pairs this alternative with a corresponding alternative in the supplementary issue based on the decreasing function $\mathbf{y}$. This means that when agreement is possible, the mechanism must always make selections from a special set of bundles. At these bundles, a more preferred alternative from the main issue must be paired with a less preferred alternative from the supplementary issue (since $\mathbf{y}$ is decreasing). We interpret these bundles as representing possible "compromises" between the two issues. As such, we henceforth call these bundles **logrolling bundles**. For a given decreasing function $\mathbf{y}$, let $\mathbf{B}^{\mathbf{y}}$ be the set of all logrolling bundles. When the main and supplementary issues have the same number of alternatives, this set is unique and $\mathbf{y}(x_k) = y_{n-k+1}$. Otherwise (i.e., when $n > m$), there can be multiple such $\mathbf{y}$'s, hence multiple classes of mechanisms.

The set $\mathbf{B}^{\mathbf{y}}$ of logrolling bundles constitutes the "backbone" of every strategy-proof, efficient, and individually rational mechanism in the sense that the diagonal of any such mechanism (i.e., when $\ell = j$, so there is a unique alternative in the zone of mutual gain) must always be comprised of logrolling bundles. The mediator has discretion over the choice of the **precedence order** $\trianglerighteq$ on $\mathbf{X}$. When the zone of mutual gain has more than one alternative (i.e., $\ell > j$), the logrolling bundle is selected according to the chosen precedence order $\trianglerighteq$. If the zone of mutual gain is empty (i.e., when $\ell < j$), then the mechanism always chooses the designated outside option ϕ.

For the rest of the paper, we refer to f as a **logrolling mechanism** if it satisfies the properties described in Theorem 1, and denote it by $f^{\trianglerighteq}$. The choice of the set of logrolling bundles together with the precedence order characterizes each mechanism. Before giving a sketch of the proof of Theorem 1, we provide an example of these mechanisms.

Example 2 (A logrolling mechanism): Suppose the main issue $\mathbf{X}$ consists of five alternatives (i.e., $m = 5$) and the supplementary issue $\mathbf{Y}$ has at least five alternatives as in Example 1. A set of logrolling bundles is then

$$\mathbf{B}^{\mathbf{y}} = \{(x_1, \mathbf{y}(x_1)), (x_2, \mathbf{y}(x_2)), (x_3, \mathbf{y}(x_3)), (x_4, \mathbf{y}(x_4)), (x_5, \mathbf{y}(x_5))\}$$

for some injective and decreasing function $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$. For notational simplicity, we let $b_k = (x_k, \mathbf{y}(x_k))$ for all $k = 1, \ldots, 5$. Let us construct the logrolling mechanism $f^{\trianglerighteq}$ associated with the precedence order $\trianglerighteq$ where

$$\trianglerighteq : x_5 \ x_1 \ x_4 \ x_2 \ x_3.$$

Suppose we would like to determine $f^{\trianglerighteq}_{3,1}$. The zone of mutual gain is $[x_1, x_3] = \{x_1, x_2, x_3\}$. The highest precedence alternative in this set is x_1. Thus, $f^{\trianglerighteq}_{3,1} = b_1$. Similarly, to determine $f^{\trianglerighteq}_{4,2}$ we maximize $\trianglerighteq$ on $[x_2, x_4] = \{x_2, x_3, x_4\}$, which yields x_4. Hence, $f^{\trianglerighteq}_{4,2} = b_4$.

Alternatively, we can start from the diagonal and let the logrolling bundles "spread" in the southwestern direction following $\trianglerighteq$. The main diagonal is filled with the members of the set of logrolling bundles, $\mathbf{B}^{\mathbf{y}}$. Namely, we have $f^{\trianglerighteq}_{1,1} = b_1$ in the first diagonal entry, $f^{\trianglerighteq}_{2,2} = b_2$ in the second diagonal entry, and so on. Since alternative x_5 has the highest precedence order, the corresponding logrolling bundle, b_5, claims all the entries to its southwest, which amounts to the set of all entries on the bottom row to the left of $f^{\trianglerighteq}_{5,5}$. The second-highest precedence belongs to x_1, and the corresponding logrolling bundle, b_1, claims all the unfilled entries to its southwest. Thus, starting from the entry $f^{\trianglerighteq}_{1,1}$ on the main diagonal, all the remaining empty entries on the first column fill up with b_1. Finally, whenever the negotiators have no mutually negotiable alternative in the main issue, the mechanism offers ϕ. The following matrix shows this logrolling mechanism $f^{\trianglerighteq}$.

	x_1^2	x_2^2	x_3^2	x_4^2	x_5^2
x_1^1	b_1	ϕ	ϕ	ϕ	ϕ
x_2^1	b_1	b_2	ϕ	ϕ	ϕ
x_3^1	b_1	b_2	b_3	ϕ	ϕ
x_4^1	b_1	b_4	b_4	b_4	ϕ
x_5^1	b_5	b_5	b_5	b_5	b_5

Figure 2: *A standard member of the logrolling mechanisms family*

Sketch of the proof of Theorem 1: The proof follows four main steps: (1) establishing an injective and decreasing function $\mathbf{y}$ from $\mathbf{X}$ to $\mathbf{Y}$, and thus the set $\mathbf{B}^{\mathbf{y}}$; (2) proving that each entry of the lower half of the matrix f comes from the set $\mathbf{B}^{\mathbf{y}}$; (3) establishing the binary relation $\trianglerighteq$ over $\mathbf{X}$ that is transitive and antisymmetric; and (4) proving that each entry of the lower half of the matrix f is in fact the maximal element of a particular subset of $\mathbf{X}$ with respect to the partial order $\trianglerighteq$.

These four steps prove particular claims by utilizing the following core idea, which we call the **weak axiom of revealed precedence (WARP)**: If two distinct alternatives x, x' in issue $\mathbf{X}$ are in the zone of mutual gain at some type profile, and f suggests a bundle with x at that profile, then it cannot be the case that f suggests x' at another type profile where both x and x' are in the zone of mutual gain. Therefore, whenever the zone of mutual gain is non-empty, a strategy-proof, efficient and individually rational mediation mechanism behaves as if it is a single valued "choice mechanism" that satisfies the weak axiom of revealed preference (see Rubinstein 2012).

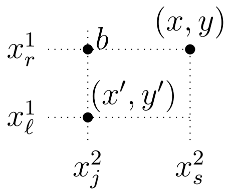


Figure 3

The intuition behind WARP is simple. Suppose it does not hold. Figure 3 indicates some entries at the lower half of the matrix f, where distinct alternatives x and x' of issue $\mathbf{X}$ are in the zone of mutual gain at all type profiles represented in this figure. Note that type x_ℓ^1 of Negotiator 1 is more accepting than type x_r^1 in the sense that all the negotiable alternatives for the latter is also negotiable for the former, and so by individual rationality and the deal-breaker property, bundle b must consist of negotiable alternatives for both types of Negotiator 1. Strategy-proofness implies that for type x_r^1, $U_1(b, x_r^1) \geq U_1(x', y', x_r^1)$ for all $U_1 \in \mathcal{D}_1^R$. The same inequality must hold for all types where x' is negotiable. (This is true because rationalizable utility functions are type invariant. Namely, for every rationalizable utility function $U \in \mathcal{D}_1^R$, there is a discernible utility function $u_1 \in \mathcal{D}_1$ such that $U(b, t_1) = u(b)$ whenever b is acceptable bundle for type t_1.) The converse of the last inequality is also true (i.e., $U_1(x', y', x_\ell^1) \geq U_1(b, x_\ell^1)$ for type x_ℓ^1 of Negotiator 1) by strategy-proofness. Because U_1 is strict, we must have $b = (x', y')$.

By repeating the symmetric arguments for Negotiator 2 and recalling that b and (x', y') are the same, we conclude that all these three bundles must be the same, contradicting our presumption that x and x' are distinct.

Individual rationality and efficiency of f, together with deal-breaker property of the rationalizable utilities, imply that for any $j, \ell \in \mathcal{I}$ with $j \leq \ell$, the alternative $f^{\mathbf{X}}_{\ell,j}$ must be an element of the non-empty set $[x_j, x_\ell]$ since it is the zone of mutual gain at type profile corresponding to the entry (ℓ, j). Therefore, an alternative $x_k \in \mathbf{X}$ must appear on the main diagonal only once (in particular $f^{\mathbf{X}}_{k,k} = x_k$) because the zone of mutual gain is singleton at this type profile. We use the bundles on the main diagonal to generate the mapping $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ by setting $\mathbf{y}\left(f^{\mathbf{X}}_{k,k}\right) = f^{\mathbf{Y}}_{k,k}$ for all $k \in \mathcal{I}$. At any entry on the second diagonal (i.e., $f_{k+1,k}$), f must offer either x_k or x_{k+1} because these are the only alternatives in the zone of mutual gain at type profile corresponding to this entry $(k+1, k)$. But in the second diagonal, a strategy-proof mechanism cannot offer the same alternative in issue $\mathbf{X}$ and a better (or worse) alternative in issue $\mathbf{Y}$. That is, any entry on the second diagonal of f must be equal to the main diagonal entry that is located either to its right or above. The last observation and strategy-proofness imply that $\mathbf{y}$ must be injective and decreasing because the second diagonal entry involves a worse alternative in the main issue for one negotiator, so she must be compensated by a better alternative in issue $\mathbf{Y}$. We denote the set of all bundles on the main diagonal by $\mathbf{B}^{\mathbf{y}}$ (Step 1). Similar arguments and WARP imply that each entry of the lower half of the matrix f is equal to an entry on the main diagonal of f (Step 2).

Much like the case in rationalizable choice mechanisms, WARP implies that f behaves as if it follows a binary relation (which we call a precedence order) over the set of alternatives in the main issue $\mathbf{X}$ in such a way that it always picks the alternative in issue $\mathbf{X}$ that is revealed to be "better" than any other alternative in the zone of mutual gain. Therefore, we construct the partial order as follows. Take any type profile (x^1_ℓ, x^2_j) that corresponds to an entry in the lower half of the matrix f and consider the set of all alternatives in the zone of mutual gain at that profile (i.e., $[x_j, x_\ell]$). We say $f^{\mathbf{X}}_{\ell,j} \trianglerighteq x$ whenever $x \in [x_j, x_\ell]$ (Step 3).

It follows from construction and WARP that the binary relation $\trianglerighteq$ is antisymmetric and transitive, and each entry of the lower half of the matrix is indeed the maximal element of the zone of mutual agreement at the type profile corresponding to that entry (Step 4). By individual rationality and the deal-breaker property of the rationalizable utilities, f must choose ϕ above the diagonal when the zone of mutual gain is empty.

3.2 Full Characterization and Quid Pro Quo

Theorem 1 provides the necessary conditions that a strategy-proof, efficient, and individually rational mediation mechanism must satisfy. However, a logrolling mechanism is not guaranteed to be strategy-proof in general. Therefore, we now search for a condition on preferences that guarantees strategy-proofness. Since the class of logrolling mechanisms contains the only candidates that can achieve the properties in Theorem 1, ensuring that a logrolling mechanism is strategy-proof automatically entails imposing a discipline on preference profiles regarding how negotiators rank the logrolling bundles. To this end, we define a key notion.

Definition 1. *A non-empty set $\mathcal{D} \subseteq \mathcal{U}$ satisfies* ***quid pro quo*** *if there exists an injective and decreasing function $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$, and a partial order $\succeq_{\mathbf{y}}$ over $\mathbf{X}$ such that*

1. *for any distinct $x, x' \in \mathbf{X}$, $x \succeq_{\mathbf{y}} x'$ whenever there exists $i \in \mathbf{I}$ such that*

(i) $u_i(x', \mathbf{y}(x)) > u_i(x, \mathbf{y}(x)) > u_i(x', \mathbf{y}(x'))$ *for all $u_i \in \mathcal{D}_i$, and*

(ii) *if there exists $y \in \mathbf{Y}$ satisfying $u_i(x', y) > u_i(x, \mathbf{y}(x))$ for all $u_i \in \mathcal{D}_i$, then $u_{-i}(x, \mathbf{y}(x)) > u_{-i}(x', y)$ for some u_{-i} satisfying $(u_i, u_{-i}) \in \mathcal{D}$.*

2. *for any non-empty zone of mutual gain S, the poset $(S, \succeq_{\mathbf{y}})$ is a join semilattice.*

Given that negotiators' utility functions satisfy quid pro quo, we let $\Pi_{\mathcal{D}}$ denote the set of all partial orders induced by the set of discernible utility profiles in $\mathcal{D}$. Namely, $\Pi_{\mathcal{D}}$ is the set of all partial orders $\succeq_{\mathbf{y}}$ over $\mathbf{X}$ such that the decreasing function $\mathbf{y}$ and $\succeq_{\mathbf{y}}$ satisfy Definition 1.

Quid pro quo property implies that negotiators are willing to make concessions in the main issue $\mathbf{X}$ for a more favorable treatment in the supplementary issue $\mathbf{Y}$. Put differently, it should be possible to find some alternatives in issue $\mathbf{Y}$ that are sufficiently attractive for at least one of the negotiators to reverse her ranking of some alternatives in the main issue when they are bundled together. Specifically, condition $(1.i)$ says that for some pairs of negotiable alternatives x, x' in the main issue, there must be a negotiator such that although she prefers x' to x, there is a pair of alternatives y, y' in the supplementary issue with the property that she prefers (x, y) to (x', y'). Such possibility of a preference reversal induces the partial order $x \succeq_{\mathbf{y}} x'$.

Condition $(1.ii)$ is a purely technical assumption, which ensures that all logrolling bundles generated by $\mathbf{y}$ are efficient. It constrains the decreasing function $\mathbf{y}$ to select a Pareto undominated alternative in issue $\mathbf{Y}$ that allows for the required preference reversal. Condition $(1.ii)$ is redundant when $|\mathbf{X}| = |\mathbf{Y}|$, and satisfied otherwise when mediation is described as a division of surplus problem (see Example 1) and negotiators have standard utility functions (see Example 3). These preference reversals define a partial order on $\mathbf{X}$ and condition (2) requires that this partial order together with any zone of mutual gain form a join semilattice.

We are now ready to provide a full characterization result.

Theorem 2. *There exists a mechanism f satisfying strategy-proofness, efficiency, and individual rationality over a non-empty set $\mathcal{D} \subseteq \mathcal{U}$ if and only if $\mathcal{D}$ satisfies quid pro quo and there is a partial order $\succeq_{\mathbf{y}} \in \Pi_{\mathcal{D}}$ such that $f = f^{\succeq_{\mathbf{y}}}$.*

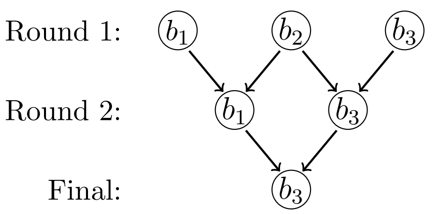
Theorem 2 states that quid pro quo is both necessary and sufficient for the existence of strategy-proof, efficient, and individual rational mediation mechanisms, and any such mechanism must be a logrolling mechanism associated with a precedence order $\succeq_{\mathbf{y}}$ induced by the domain of discernible utility profiles $\mathcal{D}$.

A Practical Depiction of Quid Pro Quo

Definition 1 expresses the quid pro quo property based on an order-theoretic semilattice structure. An equivalent and arguably more intuitive description uses a recursive and algorithmic process on the set of logrolling bundles, which we present through a simple example. This alternative structure is practically useful since it helps better understand the quid pro quo property and its role in Theorem 2. The essence of quid pro quo is that the negotiators' preferences allow an "*elimination tournament*" of the form we discus below, where there is always a winner of each match-up at each round. Each round of this tournament effectively represents the corresponding diagonal of the strategy-proof, efficient, and individual rational mechanism to be constructed.

As an example, consider the case with three alternatives in each issue. The tournament always starts with all logrolling bundles ordered from b_1 to b_3 (see the left side of Figure 4), where $b_k = (x_k, y_{4-k})$.[35] Consider the rational types of each negotiator (i.e., types x_3^1 and x_1^2). In the first round of the tournament, each logrolling bundle matches up with its adjacent neighbor (i.e., both b_1 and b_3 match only with b_2). In the match-up between b_k and b_{k+1}, the "winner" is b_{k+1} if Negotiator 1 unambiguously ranks b_{k+1} over b_k (i.e., $u_1(b_{k+1}) \geq u_1(b_k)$ for all $u_1 \in \mathcal{D}_1$). Otherwise (if Negotiator 2 unambiguously ranks b_k over b_{k+1}), the winner of this match-up is b_k. These are the conditions implied by $(1.i)$ of Definition 1. If Negotiator 1 unambiguously ranks b_{k+1} over b_k and Negotiator 2 also unambiguously ranks b_k over b_{k+1}, then both of these bundles can be the winner. In such cases, the mediator (i.e., the partial order $\succeq_{\mathbf{y}}$ that we create) has the freedom to choose either one of these two bundles to proceed to the next round.

We suppose in our example that Negotiator 2 unambiguously ranks b_1 over b_2 and Negotiator 1 unambiguously ranks b_3 over b_2. Therefore, b_1 and b_3 win over b_2 and move to the next round. In the second round, the winners of the first round (i.e., b_1 and b_3) match up (see the second raw on the left side of Figure 4). In the second round b_3 (respectively, b_1) would be the winner if Negotiator 1 (respectively, Negotiator 2) unambiguously ranks b_3 over b_1 (respectively, b_1 over b_3).



	x_1^2	x_2^2	x_3^2
x_1^1	b_1		
x_2^1	b_1	b_2	
x_3^1	b_3	b_3	b_3

Figure 4: *An example for the elimination tournament and the matrix representation for the corresponding mediation rule*

If Negotiator 1 ranks b_1 over b_3 for some $u_1 \in \mathcal{D}_1$ and Negotiator 2 ranks b_3 over b_1 for some $u_2 \in \mathcal{D}_2$, then there is no winner of this match-up and the process fails. In this case, we go back to the previous round(s). If the mediator had the freedom to choose the winner of any match-ups in the earlier rounds, then we replace the winner(s) of these match-ups and reiterate the process. If the mediator had no freedom to choose the winner in the earlier rounds, or if none of these reiterations yield a match-up in the second round with a winner, then the process fails, meaning that $\mathcal{D}$ does not satisfy quid pro quo.

[35]When $m = n$, then decreasing function $\mathbf{y}$ is unique. For cases where $m < n$, the process may start with any such $\mathbf{y}$. If the logrolling bundles in $\mathbf{B}^{\mathbf{y}}$ fail to satisfy $(1.ii)$, then $\mathbf{y}$ should be replaced and the entire process should be repeated with the new logrolling bundles.

Suppose, for the sake of the argument, Negotiator 1 unambiguously ranks b_3 over b_1 and Negotiator 2 unambiguously ranks b_1 over b_3 in our simple example. Then either bundle can be the winner of the second round. In the illustration above, we have illustrated b_3 as the winner of the tournament.

The match-up configurations in this entire tournament is in fact the join semilattice structure implied by condition (2) of Definition 1. Theorem 2 says that we can use this tournament structure in creating logrolling mechanisms that are strategy-proof, efficient, and individually rational over $\mathcal{D}$. The winners of each round fill up the corresponding diagonals. For the tournament described above, the order of the logrolling bundles in the first round gives the placement order of these bundles (from the top corner to the bottom corner) along the first diagonal, the order in the second round gives the placement order along the second diagonal, and the last winner, b_3, fills up the bottom left entry of this matrix (which is the last diagonal). The constructed mechanism corresponds to the logrolling mechanism $f^{\succeq_\mathbf{y}}$ with $\succeq_\mathbf{y}$: x_3 x_1 x_2. Recall that both b_1 and b_3 were winners in the second round in our example, so the negotiators' preferences (i.e., $\mathcal{D}$) also admit a second logrolling mechanism $f^{\succeq'_\mathbf{y}}$ with $\succeq'_\mathbf{y}$: x_1 x_3 x_2.

Many standard utility functions are compatible with the quid pro quo condition, and this is illustrated in the next example.

Example 3 (Quid Pro Quo Under Standard Preferences): We re-consider the utility functions u_1 and u_2 in Example 1 and discuss various sufficiency conditions that guarantee the quid pro quo property, and hence strategy-proofness. As a first scenario, suppose that Negotiator 1's utility function u_1 satisfies the following: There is a decreasing one-to-one function $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ such that for any $x_k, x_j \in \mathbf{X}$, where $j > k$, the following condition holds:

$$\frac{x_k^{\alpha_1} - x_j^{\alpha_1}}{\mathbf{y}(x_j) - \mathbf{y}(x_k)} < 1. \tag{5}$$

The inequality system (5) implies a particular type of preference reversals, where for all $x_k, x_{k+1} \in \mathbf{X}$, $u_1(x_{k+1}, \mathbf{y}(x_{k+1})) > u_1(x_k, \mathbf{y}(x_k))$. In this case, the utility function u_1 implies the quid pro quo property without any restrictions on u_2. The partial order satisfying $x_{k+1} \succeq_\mathbf{y} x_k$ for $k = 1, \ldots m-1$ and $\mathbf{y}$ satisfy Definition 1. The logrolling mechanism induced by $\succeq_\mathbf{y}$ (which is what we refer as the Negotiator 1-optimal mechanism in Section 4) is strategy proof, efficient, and individually rational.

Adding restrictions also on u_2 would give rise to other strategy-proof mechanisms from the logrolling family. For example, consider a second scenario in which the analogous inequality system to (5) also holds for u_2. Then all partial orders on $\mathbf{X}$ and $\mathbf{y}$ satisfy Definition 1, and thus all members of the logrolling family are strategy-proof. On the other hand, there are also several types of preference reversals between these two scenarios that would also ensure the quid pro quo property. Such "intermediate" scenarios would have only a subset of the inequalities in (5) holding together with a subset of the analogous inequality system to (5) for u_2.

A closer look at inequality (5) provides some valuable insights that can be generalized to any additively separable utility function. The ratio in (5) represents the *marginal rate of ranking* *substitution* with respect to the decreasing function $\mathbf{y}$, which measures the rate of substitution (in utility levels) Negotiator 1 needs to keep the combined total ranking of two logrolling bundles the same while maintaining the desired preference reversal. To see this, note that the total ranking of any logrolling bundle is $m+1$.[36] Moving from the bundle $(x_j, \mathbf{y}(x_j))$ to the bundle $(x_k, \mathbf{y}(x_j))$ increases the total ranking of the outcome by $j-k$ positions in favor of Negotiator 1, and the nominator measures the additional utility she enjoys from this change. Next, moving from bundle $(x_k, \mathbf{y}(x_j))$ to $(x_k, \mathbf{y}(x_k))$ decreases the total ranking of the outcome by $j-k$ positions for Negotiator 1, and the denominator measures the utility loss she suffers from this change. Therefore, the ratio in (5) measures the rate of utility substitution Negotiator 1 needs between the alternatives in the main and the supplementary issue to keep the total ranking fixed.[37]

For concreteness, consider the following simple numerical example: Suppose that $\bar{x} = 10$, $\bar{y} = 6$, $\mathbf{X} = \{1, 3, 5, 7, 9\}$ and $\mathbf{y}(x) = \frac{11-x}{2}$, so that the set of logrolling bundles is $\mathbf{B}^{\mathbf{y}} = \{(1, 5), (3, 4), (5, 3), (7, 2), (9, 1)\}$. Given the utility functions u_1 and u_2 in Example 1, quid pro quo is satisfied when $\alpha_i \in (0, \frac{3}{5}]$ for each $i \in \mathbf{I}$.

Indeed, while Negotiator 1 prefers higher values of x, she prefers $(x, \mathbf{y}(x))$ over $(x', \mathbf{y}(x'))$ for all $x, x' \in \mathbf{X}$ with $x' > x$. Based solely on 1's preferences, this induces a complete order on $\mathbf{X}$ where $x \succeq^1_{\mathbf{y}} x'$ when $x' > x$. Similarly, Negotiator 2's preferences over bundles satisfies an analogous reversal. Based solely on 2's preferences, this induces a complete order on X where $x \succeq^2_{\mathbf{y}} x'$ when $x > x'$. Consequently, any linear order on $\mathbf{X}$ satisfies Condition $(1.i)$ of Definition 1. Indeed, any logrolling mechanism associated with $\mathbf{B}^{\mathbf{y}}$ and any well-defined partial order on $\mathbf{X}$ is strategy-proof under these preferences.[38]

Sketch of the Proof of Theorem 2

Consider the "if" part. Mechanism $f^{\succeq_{\mathbf{y}}}$ satisfies individual rationality because it never suggests a non-negotiable alternative or unacceptable outcome. For efficiency, consider a bundle $b = (x_b, \mathbf{y}(x_b))$ suggested by $f^{\succeq_{\mathbf{y}}}$ at some type profile $t \in \mathbf{T}$. Suppose for a contradiction that another bundle $a = (x_a, \mathbf{y}(x_a)) \in \mathbf{B}^{\mathbf{y}}$, where x_a is also in the zone of mutual gain at this type profile t, Pareto dominates b. Because $f^{\succeq_{\mathbf{y}}}$ suggests b while both x_a and x_b are available, we must have $x_b \succeq_{\mathbf{y}} x_a$. Moreover, because negotiators' utility functions satisfy quid pro quo, there is a negotiator i such that for all $u_i \in \mathcal{D}_i$, $u_i(x_a, y) > u_i(x_b, y)$ for all $y \in Y$ while $u_i(b) > u_i(a)$, contradicting that bundle a Pareto dominates b (the strict inequalities follow from the fact that

[36] Consider a logrolling bundle $(x_k, \mathbf{y}(x_k))$: x_k is the k^{th}-best alternative for Negotiator 1 in the main issue, and $\mathbf{y}(x_k)$ is the $m+1-k^{th}$-best alternative—within the range of $\mathbf{y}$—in the supplementary issue.

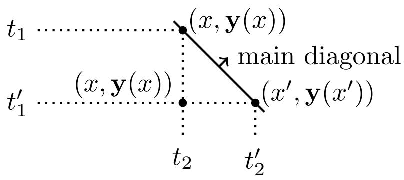
[37] The ratio in (5) being less than 1 does not imply that the supplementary issue $\mathbf{Y}$ is more "important" for Negotiator 1 than the main issue $\mathbf{X}$. Rather, it represents the relative rate of differences in utility. This condition holds, for example, if the utility levels of the alternatives in $\mathbf{X}$ are "closer" to each other relative to those in $\mathbf{Y}$ although alternatives in $\mathbf{X}$ correspond to higher utility levels than those in $\mathbf{Y}$.

[38] For the Cobb-Douglas specification given by utility functions v_1 and v_2 in Example 1, quid pro quo is satisfied if $\frac{a_i}{b_i} \in [\frac{7}{12}, 1]$ for each $i \in \mathbf{I}$. Although negotiator 1 ceteris paribus prefers higher values of x, it can be verified that $v_1(5, 3) > v_1(7, 2) > v_1(9, 1)$ for negotiable alternatives. This induces the partial order $\succeq_{\mathbf{y}}$ such that $5 \succeq_{\mathbf{y}} 7 \succeq_{\mathbf{y}} 9$. Similarly, although negotiator 2 ceteris paribus prefers lower values of x (getting more of asset $\mathbf{X}$), we have $v_2(5, 3) > v_2(3, 4) > v_2(1, 5)$ for negotiable alternatives. This induces the partial order $\succeq_{\mathbf{y}}$ such that $5 \succeq_{\mathbf{y}} 3 \succeq_{y} 1$. A logrolling mechanism associated with $\mathbf{B}^{\mathbf{y}}$ and partial order $\succeq_{\mathbf{y}}$ is strategy-proof. Figure 7 illustrates one such mechanism.

u_i's are strict and $a \neq b$). If $a \notin \mathbf{B}^{\mathbf{y}}$, then by condition (i.2) of Definition 1, there is no alternative $y \in \mathbf{Y}$ that can be paired with x_a so that bundle a Pareto dominates b. Hence, f must be efficient.

Regarding strategy-proofness of f, a profitable deviation is never possible, by the deal-breaker property, from or to a type profile in which $f^{\succeq_{\mathbf{y}}}$ suggests ϕ. Therefore, consider a type profile where $f^{\succeq_{\mathbf{y}}}$ suggests a bundle $b' = (x_{b'}, \mathbf{y}(x_{b'})) \neq \phi$. Any deviation of, say, Negotiator 1 to a less-accepting type to get $a' = (x_{a'}, \mathbf{y}(x_{a'})) \neq b'$, which must be located on the same column with b' (since Negotiator 2's type is fixed) but on a lower row (since Negotiator 1 is deviating to a less-accepting type to get a'), is never profitable: This is true because (1) we must have $x_{b'} \succeq_{\mathbf{y}} x_{a'}$ since the logrolling mechanism $f^{\succeq_{\mathbf{y}}}$ suggests b' when both $x_{a'}$ and $x_{b'}$ are in the zone of mutual gain; (2) bundle a' must be appearing on a lower row on the main diagonal of $f^{\succeq_{\mathbf{y}}}$ than b' does because $\succeq_{\mathbf{y}}$ is transitive; and thus (3) it must be the case that Negotiator 1 prefers alternative $x_{a'}$ over $x_{b'}$ (and Negotiator 2 prefers $x_{b'}$ over $x_{a'}$) whenever both these alternatives are negotiable for her, and so, Negotiator 1 must prefer bundle b' over a' because preferences satisfy quid pro quo and $x_{b'} \succeq_{\mathbf{y}} x_{a'}$. Similar reasoning proves that Negotiator 1 has no incentive to deviate to a more-accepting type. Hence, $f^{\succeq_{\mathbf{y}}}$ is strategy-proof.

Consider now the "only if" part. By Theorem 1, strategy-proofness, efficiency, and individual rationality of f over some non-empty subset $\mathcal{D}$ of $\mathcal{U}$ imply an injective and decreasing function $\mathbf{y}$ and a partial order $\succeq_{\mathbf{y}}$ such that $f = f^{\succeq_{\mathbf{y}}}$. To prove $\succeq_{\mathbf{y}} \in \Pi_{\mathcal{D}}$, and so $\mathcal{D}$ satisfies quid pro quo, we need to show that $\succeq_{\mathbf{y}}$ and $\mathbf{y}$ satisfy Definition 1. Condition (1.ii) is simply implied by the efficiency of f over $\mathcal{D}$. For condition (1.i) take any x, x' with $x \succeq_{\mathbf{y}} x'$. By the construction of $\succeq_{\mathbf{y}}$ in the proof of Theorem 1, we know that $x \succeq_{\mathbf{y}} x'$ implies that f must be suggesting a bundle with x at some type profile where both x and x' are in the zone of mutual gain (e.g., (t_1', t_2), see the figure below). Assuming, without loss of generality, that for some $y \in Y$, $u_1(x, y) \geq u_1(x', y)$ for all $u_1 \in \mathcal{D}_1$, Negotiator 2 has $u_2(x, y) \leq u_2(x', y)$ for all $u_2 \in \mathcal{D}_2$ because her preferences over individual issues are diametrically opposed. It is easy to verify that strategy-proofness of f implies $u_2(x, \mathbf{y}(x)) \geq u_2(x', \mathbf{y}(x'))$ because otherwise type t_2 would deviate to t_2'. The last inequality is what condition (1.i) requires.



Finally, the collection of sets $[x_j, x_\ell]$ where $1 \leq j \leq \ell \leq m$ constitutes the set of all non-empty zones of mutual gain, and every doubleton $\{x, x'\} \subseteq [x_j, x_\ell]$ has a least upper bound in $[x_j, x_\ell]$, which is $x^*_{[x_j, x_\ell]}$. Thus, $(S, \trianglerighteq)$ is a semilattice for any non-empty zone of mutual gain, as required by condition (2) of Definition 1.

3.3 A Visual Characterization of the Class of Logrolling mechanisms

To provide further insight into the logrolling mechanisms that are characterized by Theorems 1 and 2, we offer a geometric analysis of these mechanisms. We first take a mechanism $f =$

$[f_{\ell,j}]_{(\ell,j)\in\mathcal{I}^2}$ and introduce a couple of definitions to represent different rectangular and triangular regions of this matrix. In the following two definitions we slightly abuse notation and terminology in order to keep track of the entries contained in a rectangular/triangular region. Namely, we use $f_{\ell,j}$ to refer to entry (ℓ, j) of the matrix rather than the specific bundle that mechanism f assigns to that entry.

Definition 2. *Consider the entry $f_{k,k}$ for some $k \in \mathcal{I}$ and an entry that lies (weakly) to its southwest, $f_{\ell,j}$ with $1 \leq j \leq k \leq \ell \leq m$. The* ***rectangle*** *induced by $f_{k,k}$ and $f_{\ell,j}$, denoted by $\square^k_{\ell,j}$, is the set of all entries in the rectangular region of the matrix (inclusively) enveloped between rows k and ℓ and columns k and j. Namely, $\square^k_{\ell,j} = \bigcup_{\substack{j\leq s\leq k \\ k\leq t\leq \ell}} \{f_{t,s}\}$.*

Definition 3. *The* ***triangle*** *induced by an entry $f_{\ell,j}$ with $1 \leq j \leq \ell \leq m$, denoted by $\triangle_{\ell,j}$, is the set of all entries in the triangular region of the matrix that is (inclusively) enveloped by the entry $f_{\ell,j}$, row ℓ, column j, and the main diagonal. Namely, $\triangle_{\ell,j} = \bigcup_{j\leq k\leq \ell} \{f_{k,j}, f_{k,j+1}, ..., f_{k,k}\}$.*

A rectangle/triangle is merely a collection of entries of the matrix induced by mechanism f (i.e., sets of pairs of indexes). Note that an entry on the main diagonal is a special triangle (and also a special rectangle) that consists of a singleton entry. Furthermore, the entire main diagonal of the matrix and all the entries to its southwest constitute the largest possible triangle $\triangle_{m,1}$. Given a triangle $\triangle_{\ell,j}$, its entries that lie on the main diagonal are said to be on the *hypotenuse* of $\triangle_{\ell,j}$. A partition of the lower half of the matrix is called a *rectangular (triangular) partition* if and only if it is the union of disjoint rectangles (triangles).[39]

Theorem 3 (Visual Characterization). *Let $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ be an injective and decreasing function and $\succeq_{\mathbf{y}}$ be a partial order on $\mathbf{X}$. The following statements are equivalent.*

(i) f is equivalent to a logrolling mechanism, namely $f(t) = f^{\succeq_{\mathbf{y}}}(t)$, at all type profiles $t \in \mathbf{T}$ where the zone of mutual gain is non-empty.

(ii) The triangle $\triangle_{m,1}$ has a rectangular partition such that f assigns a unique bundle from the set of logrolling bundles $\mathbf{B}^{\mathbf{y}}$ to each rectangle in this partition.[40]

Part *(ii)* of Theorem 3 states that a logrolling mechanism f can be represented as the union of m disjoint rectangular regions. Each rectangle has a distinct corner entry on the main diagonal that contains the logrolling bundle that fills up the entire rectangle. Procedurally, these rectangles are obtained as follows. Given the precedence order $\succeq_{\mathbf{y}}$ on $\mathbf{X}$, start with the logrolling bundle with the highest-precedence alternative (i.e., highest-precedence bundle). Starting from the entry of this bundle on the hypotenuse of the largest triangle, $\triangle_{m,1}$, let it fill up all the entries located to its southwest. This creates the first and largest rectangle $\square$, and leads to a triangular

[39]Note that a rectangular partition consists of m disjoint rectangles. For example, $\{\square^k_{k,1}\}^m_{k=1}$ and $\{\square^k_{m,k}\}^m_{k=1}$ are two obvious rectangular partitions of $\triangle_{m,1}$. These two partitions correspond respectively to what we will later refer to as the Negotiator 1- and Negotiator 2-optimal mechanisms.

[40]More formally, for any $\square$ in the partition of $\triangle_{m,1}$ and any bundles $b, b' \in \square$, $b = b'$; but for any distinct pair $\square, \square'$ in the partition of $\triangle_{m,1}$, $b \in \square$ and $b' \in \square'$ implies $b \neq b'$.

partition of $\triangle_{m,1} \backslash \square$. Next, pick any triangle from this partition and let the highest-precedence bundle on the hypotenuse of this triangle fill up all the entries that are located to its southwest. This leads to a second rectangle $\square'$ as well as a unique triangular partition of $\triangle_{m,1} \backslash \{\square, \square'\}$. The process can be iterated in this fashion until the entire triangle $\triangle_{m,1}$ is partitioned into m disjoint rectangles in m steps. Figure 5a provides an illustration of one such partitioning, where $b_k = (x_k, \mathbf{y}(x_k))$ for $k = 1, ..., 9$. This process effectively traces the semilattice $(\mathbf{X}, \succeq_{\mathbf{y}})$ in Figure 5b. Conversely, any such geometric set, namely any rectangular partition of $\triangle_{m,1}$, can be used to construct a precedence order and a corresponding logrolling mechanism.

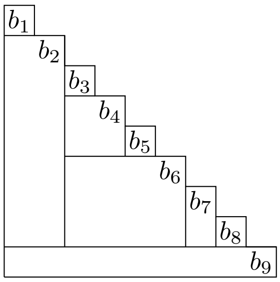


Figure 5a: *A rectangular partitioning of* $f^{\succeq_{\mathbf{y}}}$ *with m=9*

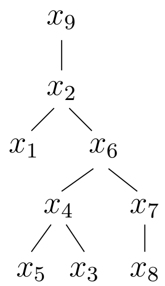


Figure 5b: *A semilattice* $(\mathbf{X}, \succeq_{\mathbf{y}})$

3.4 A Practical Formulation of Logrolling Mechanisms

As briefly discussed in Section 1.1, dispute resolution protocols of several ODR platforms such as the SquareTrade operate by generating a menu of offers from which negotiators are asked to make selections sequentially. In light of Theorem 3, the working principles of the logrolling mechanisms lead to a similar interpretation that is also reminiscent of the divide-and-choose mechanisms in fair division literature.

In particular, a logrolling mechanism can be thought to work as a "*shortlisting mechanism*" in a decentralized fashion: One negotiator offers a shortlist of bundles as negotiable solutions for the dispute, the mediator communicates these options to the other negotiator who then chooses her favorite bundle from this list. To see this, observe that when Negotiator 1 reports her type as x^1_ℓ, it can be viewed as Negotiator 1 forming a shortlist consisting of all the bundles on row ℓ. When faced with the list of bundles Negotiator 1 offers, Negotiator 2 indeed picks the bundle $f_{\ell,j}$ since it is her favorite negotiable bundle on row ℓ by strategy-proofness. If the roles of the negotiators in this procedure were reversed, then the outcome would still be the same by symmetric arguments.[41]

For a more specific example, consider the logrolling mechanism depicted in Figure 5a. Suppose that negotiator 1 is of type x^1_3. Then we can think of her as proposing the shortlist $\{b_2, b_3, \phi\}$ to the other negotiator when she truthfully declares her type. The corresponding shortlists for other announcements as x^1_5 and x^1_7 are $\{b_2, b_4, b_5, \phi\}$ and $\{b_2, b_6, b_7, \phi\}$, respectively.

Under this interpretation, a logrolling mechanism specifies a set of shortlisted bundles that a negotiator can offer to the other party for each possible type she reports. By reporting a more-accepting type, the proposer may add new bundles or remove some from her shortlist.

[41]One drawback of the divide-and-choose rule in the context of fair division is that its outcome depends on the order of agents. Divide-and-choose also violates strategy-proofness unlike a logrolling mechanism.

Theorem 3 implies that as negotiators declare more-accepting types, suggested shortlists must satisfy some kind of regularity in the sense that a previously removed bundle can never be added back to the shortlist. For the logrolling mechanism depicted in Figure 5a, for instance, if negotiator 1 switches from x_3^1 to x_5^1, she adds bundles b_4 and b_5 to the shortlist and removes b_3. If she switches from x_5^1 to x_7^1, then she adds b_6 and b_7 and removes b_4 and b_5 from the shortlist. Note for this logrolling mechanism that once bundles b_3, b_4 or b_5 are removed, they are never added back in.

4 Special members of the logrolling family

We next visit interesting members of the logrolling family. At the outset we assume that preference domain is such that all members of the family are strategy-proof (e.g., negotiators have quasi-linear preferences). Specifically, we assume throughout this section that quid pro quo is satisfied in the following stronger sense:

Assumption 1. *The domain of preferences $\mathcal{D} \subset \mathcal{U}$ admits an injective and decreasing function $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ such that for all $i \in \mathbf{I}$ and all $x, x' \in \mathbf{X}$, if $u_i(x', y) \geq u_i(x, y)$ for some $y \in \mathbf{Y}$, then $u_i(x, \mathbf{y}(x)) \geq u_i(x', \mathbf{y}(x'))$ for all $u_i \in \mathcal{D}_i$.*

Under this assumption, any partial order $\succeq_{\mathbf{y}}$ on $\mathbf{X}$ satisfies Definition 1, so by Theorem 2, all members of the logrolling family (and only these mediation mechanisms) are strategy-proof, efficient and individually rational. Therefore, Assumption 1 renders a more meaningful comparison of the members of the logrolling family.

Three notable members are worth pointing out. A **negotiator-optimal mechanism** represents a situation of extreme partiality to one side of the dispute and is constructed by using the precedence order implied by a negotiator's preferences over the logrolling bundles. Specifically, the Negotiator 1-optimal mechanism takes

$$\succeq_{\mathbf{y}}^1 : x_m \succeq_{\mathbf{y}}^1 x_{m-1} \succeq_{\mathbf{y}}^1 \ldots \succeq_{\mathbf{y}}^1 x_1,$$

whereas the Negotiator 2-optimal mechanism takes

$$\succeq_{\mathbf{y}}^2 : x_1 \succeq_{\mathbf{y}}^2 x_2 \succeq_{\mathbf{y}}^2 \ldots \succeq_{\mathbf{y}}^2 x_m.$$

In case of severe disagreement (i.e., when the zone of mutual gain is empty), the outside option, ϕ, is the outcome. The two dual mechanisms are shown below for the case of $m = n = 5$.

	x_1^2	x_2^2	x_3^2	x_4^2	x_5^2
x_1^1	b_1	ϕ	ϕ	ϕ	ϕ
x_2^1	b_2	b_2	ϕ	ϕ	ϕ
x_3^1	b_3	b_3	b_3	ϕ	ϕ
x_4^1	b_4	b_4	b_4	b_4	ϕ
x_5^1	b_5	b_5	b_5	b_5	b_5

Figure 6a: *Negotiator 1-optimal mechanism*

	x_1^2	x_2^2	x_3^2	x_4^2	x_5^2
x_5^1	b_1	ϕ	ϕ	ϕ	ϕ
x_2^1	b_1	b_2	ϕ	ϕ	ϕ
x_3^1	b_1	b_2	b_3	ϕ	ϕ
x_4^1	b_1	b_2	b_3	b_4	ϕ
x_5^1	b_1	b_2	b_3	b_4	b_5

Figure 6b: *Negotiator 2-optimal mechanism*

A negotiator-optimal mechanism always chooses the corresponding negotiator's most-preferred bundle among the acceptable logrolling bundles. The analogous shortlisting mechanism is rather simple: Favored negotiator's shortlist includes only two bundles, which are her favorite acceptable logrolling bundle and the outside option.[42] Clearly, these two polar members of the family of logrolling mechanisms are highly unattractive in practice.[43] Fortunately, there is a remarkable member of this family that treats negotiators symmetrically.

Impartiality entails focusing on a central element of the set of logrolling bundles as a compromise. It is then intuitive for the mediator to recommend a *median* logrolling bundle when it is mutually negotiable, or seek a bundle as close to it as possible when it is not. Within the family of logrolling mechanisms, this is achieved simply by assigning the highest precedence to a median logrolling bundle, and the next precedence to those bundles that are closest to the chosen median, and so on, and the lowest precedence to the extremal logrolling bundles. This motivates the following type of mechanism, which we call a **constrained shortlisting (CS) mechanism**.

Definition 4. *Let $k \in \{\overline{k}, \underline{k}\}$ be the index of a median alternative in the main issue, where $\overline{k} = \lceil \frac{m+1}{2} \rceil$ and $\underline{k} = \lfloor \frac{m+1}{2} \rfloor$. A mechanism is a constrained shortlisting mechanism, denoted $f^{CS} = [f_{\ell,j}]_{(\ell,j)\in\mathcal{I}^2}$, if it is a logrolling mechanism that is associated with a precedence order $\succeq_{\mathbf{y}}^{CS}$, where $x_k \succeq_{\mathbf{y}}^{CS} x_{k-1} \succeq_{\mathbf{y}}^{CS} \ldots \succeq_{\mathbf{y}}^{CS} x_1$ and $x_k \succeq_{\mathbf{y}}^{CS} x_{k+1} \succeq_{\mathbf{y}}^{CS} \ldots \succeq_{\mathbf{y}}^{CS} x_m$, and $f_{\ell,j}^{CS} = \phi$ whenever $\ell < j$.*

When the number of alternatives in the main issue is odd, there is a unique constrained shortlisting mechanism When the number of alternatives is even, however, a constrained shortlisting mechanism prescribes two possible types of outcomes.[44] Figure 7 illustrates the constrained shortlisting mechanism for the case of $m = n = 5$.

When the number of alternatives is odd, the CS mechanism is a symmetric member of the logrolling mechanisms family.[45] In the lower half of the matrix, it acts as a negotiator-optimal mechanism whenever the median alternative in the main issue is not a member of the zone of mutual gain, and recommends the median logrolling bundle whenever the zone of mutual gain includes the median alternative. In other words, when both negotiators find at least half of the alternatives in the main issue negotiable, the mechanism chooses the median logrolling bundle; and, when one negotiator finds at least half of the alternatives negotiable while the other finds less than half of the alternatives negotiable, the mechanism chooses the less-accepting negotiator's favorite acceptable logrolling bundle.

[42] Alternatively, non-favored negotiator's shortlist includes all of her acceptable logrolling bundles and the outside option.

[43] Note that despite their polarity, these mechanisms are not dictatorial. Unlike a dictatorship, they remain individually rational and never get vetoed in equilibrium. Nevertheless, they hint at the possibility of the mediator having the power to tilt the balance in a dispute despite using a mechanism that meets our desiderata (i.e., efficiency, individual rationality, and strategy-proofness).

[44] In this case, the mechanism depends on whether $x_{\overline{k}}$ or $x_{\underline{k}}$ has the highest precedence.

[45] When the number of alternatives is even, no logrolling mechanism is fully symmetric.

	x_1^2	x_2^2	x_3^2	x_4^2	x_5^2
x_1^1	b_1	ϕ	ϕ	ϕ	ϕ
x_2^1	b_2	b_2	ϕ	ϕ	ϕ
x_3^1	b_3	b_3	b_3	ϕ	ϕ
x_4^1	b_3	b_3	b_3	b_4	ϕ
x_5^1	b_3	b_3	b_3	b_4	b_5

Figure 7: *Constrained shortlisting mechanism*

In discrete resource allocation problems where agents are endowed with ordinal preference rankings, fairness properties (together with efficiency) have often proved difficult to attain in the absence of monetary transfers or a randomization device. It is nevertheless worthwhile to investigate whether it is possible for a member of the logrolling mechanisms family to achieve alternative fairness requirements beyond symmetry. We next formulate one such ordinal fairness notion as a normative requirement for our context.

Given the negotiators' preferences over alternatives (not including the outside option), let $r_i(z) \in \mathcal{I}$ denote negotiator i's ranking of a negotiable alternative $z \in \mathbf{Z} \in \{\mathbf{X}, \mathbf{Y}\}$. For a normalization, we re-assign ranks 1 through m for the chosen alternatives in $\mathbf{Y}$ (i.e., those alternatives that are in the range of $\mathbf{y}$) and set the ranking of the outside option to be zero.[46] Given the logrolling mechanism $f = [f_{\ell,j}]_{(\ell,j)\in\mathcal{I}^2}$, the *rank variance of the bundle* $f_{\ell,j}$ is defined as[47]

$$var(f_{\ell,j}) \equiv \sum_{i\in\mathbf{N}} \left(r_i(f_{\ell j}^{\mathbf{X}})\right)^2 + \left(r_i(f_{\ell j}^{\mathbf{Y}})\right)^2.$$

Then, the **rank variance** of a mechanism f is the total sum of the rank variance of all possible outcomes of f, and defined as

$$Var(f) \equiv \sum_{\ell=1}^{m}\sum_{j=1}^{m} var(f_{\ell,j}).$$

Intuitively, the larger the differences between the two negotiators' rankings of the alternatives in a given bundle, the higher is the rank variance of that bundle. For example, while never recommended by a logrolling mechanism, the bundles (x_1, y_1) and (x_m, y_m) have the highest rank variance. Despite making one negotiator as well off as possible, they make the opposite negotiator as worse off as possible. In this sense, the larger the rank variance of a mediation mechanism, the more skewed it is toward extremal bundles.

Theorem 4. *A mediation mechanism minimizes rank variance within the class of logrolling mechanisms if and only if it is a constrained shortlisting mechanism.*

[46] This normalization is clearly not without loss, but simplifies the notation significantly as it treats the supplementary issue $\mathbf{Y}$ as though it also has m alternatives. Nevertheless, the rank minimizing logrolling mechanism in the absence of this normalization is merely a "shifted" version of a CS mechanism where the magnitude of the shift depends on function $\mathbf{y}$.

[47] This formulation assigns equal weights to both issues. One may also consider assigning different weights to different issues. Theorem 4 remains unchanged in that case due to the symmetric structure of the logrolling bundles under the normalization above.

5 Discussion and Extensions

In this section we provide a general discussion of our main model in light of the results obtained so far. To this end, first, we elaborate on some of our essential modeling assumptions, discuss the role they play in driving the positive results of our paper, and offer directions in which they can be extended to cases not covered in the main exposition. Second, drawing on our findings, we consider how one can go about formulating the mediation problem in a standard Bayesian setting such as that of Myerson and Sattertwaite (1983) [henceforth MS] and offer a reconciliation of the possibility results in our setup with the impossibility result in the MS setting.

5.1 Single Issue Mediation

The presence of the second issue is key for our possibility results, and it is easy to prove that there is no strategy-proof, efficient, and individually rational mediation mechanism in a single-issue mediation problem. In a simplest possible form, consider the following example: There is a single issue, $\mathbf{X}$, which has two available alternatives, x_1 and x_2, so $\mathbf{X} = \{x_1, x_2\}$. Therefore, the set of all outcomes is $\mathbf{B} = \{x_1, x_2, \phi\}$. It is public information that Negotiator 1 prefers alternative x_1 to x_2 and Negotiator 2 prefers x_2 to x_1. The ranking of the outside option, ϕ, is each negotiator's private information and each negotiator ranks any non-negotiable deal-breaker (or unacceptable) alternative below the outside option. Each negotiator has only two types (i.e., $\mathbf{T}_i = \{x_1^i, x_2^i\}$ for $i = 1, 2$), and their preferences/rankings over the outcomes are as follows:

$$\begin{array}{lll} \textit{Negotiator 1}: & x_1^1 : x_1\ \phi\ x_2 \text{ and } x_2^1 : x_1\ x_2\ \phi \\ \textit{Negotiator 2}: & x_2^2 : x_2\ \phi\ x_1 \text{ and } x_1^2 : x_2\ x_1\ \phi. \end{array}$$

Therefore, x_2^1 (respectively, x_1^2) represents the rational type of Negotiator 1 (respectively, Negotiator 2). A mechanism $f : \mathbf{T_1} \times \mathbf{T_2} \to \mathbf{B}$ can also be represented by the following matrix:

	x_1^2	x_2^2
x_1^1	$f_{1,1}$	$f_{1,2}$
x_2^1	$f_{2,1}$	$f_{2,2}$

where $f_{\ell,j} \in \mathbf{B}$ for all $\ell, j \in \{1, 2\}$.

Individual rationality requires $f_{1,2} = \phi$. Efficiency and individual rationality imply $f_{k,k} = x_k$ for $k = 1, 2$ and $f_{2,1} \in \{x_1, x_2\}$. Therefore, there are only two (deterministic) mechanisms satisfying individual rationality and efficiency in this simple framework. However, neither of these mechanisms is immune to strategic manipulation. To see this point, suppose that $f_{2,1} = x_1$. In this case, type x_1^2 of Negotiator 2 would misreport her type when Negotiator 1 is of type x_2^1 because she guarantees her favorite outcome x_2 by reporting x_2^2 instead. Symmetrically, if $f_{2,1} = x_2$, then type x_2^1 of negotiator 1 would lie about her type.[48] It is straightforward to extend this impossibility to the case with more than two alternatives.

[48] This impossibility also prevails when we allow stochastic mechanisms. In that case, the only difference in the argument would be that $f_{2,1}$ is a lottery over x_1 and x_2. However, the above deviations would still remain profitable.

5.2 Modeling conflicting preferences

Diametrically opposed preferences in each issue is without loss of generality. When describing a dispute, using diametrically opposed preferences over alternatives is intuitive. However, it is conceivable that many other situations, where preferences are not necessarily diametrically opposed, could also depict a dispute. Consider, for example, a case where the set of available alternatives is $\mathbf{X} = \{x_1, x_2, x_3, x_4, x_5\}$ and the negotiators' preferences are as follows:

$$\text{Negotiator 1: } x_1 \ x_2 \ x_3 \ x_4 \ x_5$$
$$\text{Negotiator 2: } x_3 \ x_5 \ x_4 \ x_2 \ x_1$$

These preferences are not diametrically opposed, but they are certainly conflicting to some extent as the agents cannot agree on their best alternative. Notice, however, that alternatives x_4 and x_5 are (Pareto) dominated by x_3. So, if negotiators preferences over bundles are monotonic and selecting an efficient outcome by the mediation protocol is desired, then the presence of these two alternatives is irrelevant for the problem and can be eliminated from the preferences. Thus, this particular dispute problem can be transformed into a reduced problem where the only available alternatives are x_1, x_2, and x_3 and the negotiators' preferences over these three are diametrically opposed. This observation easily generalizes to any set of alternatives, any preference profile, and to multi issue mediation problems (under monotonic preferences over bundles). Namely, eliminating inefficient alternatives yields a reduced dispute with diametrically opposed preferences.

5.3 Discrete Alternatives

Matsuo (1989) shows that it is possible to overcome the impossibility in the bilateral exchange model of MS by restricting to a finite set of types. The finiteness of the set of alternatives in each issue is a simplifying assumption and causes no loss of generality. Indeed, our results are not driven by the finiteness of the number of types. In fact, it is possible to extend the characterization of the class of logrolling rules to a continuous analogue of our model,[49] where each issue is a compact and convex subset of $\mathbb{R}$.

5.4 More than two issues or negotiators

Our two-issue model with a single main issue and a single supplementary issue is without loss of generality. If there are more than two issues in the dispute, then we can regroup these issues under two types of categories depending on whether an issue has certain or uncertain gains from mediation. In particular, let category-$\mathbf{X}$ be the collection of issues that exhibit uncertain gains from mediation (i.e., non-negotiable alternatives are boundedly rational negotiators' private information), and category-$\mathbf{Y}$ be the collection of issues that exhibit certain gains (i.e., it is common knowledge that all alternatives in these issues are negotiable for all types). Under this regrouping, each negotiator now faces a vector of alternatives for each category. The negotiators'

[49] A version of our main results for this case is available upon request.

preferences over these vectors (of alternatives) need not be diametrically opposed in general. However, as long as the negotiators' preferences are strict and monotonic, by applying the transformation discussed in Section 5.2, we can eliminate all inefficient vectors. This brings us back to an environment analogous to our main model, in which preferences over vectors are diametrically opposed.

When there are multiple parties involved in a dispute, as it would be the case for community/public disputes, we can similarly regroup them to be represented by either negotiator, effectively treating them as clones of the two negotiators. Nevertheless, there might be cases where negotiators' preferences are extremely disperse, and so grouping them into two "representative" agents is not feasible. Such mediation environments are both practically and theoretically more complex than the one we study here and in need of further research.

5.5 Reconciliation with the negative results in Bayesian settings

The influential work of MS is an important milestone in showing the difficulty of efficient trade in bargaining problems with asymmetric information. It is useful to discuss the underlying factors that are absent in the MS model, which may account for the possibility results in our model under dominant strategies. In a nutshell, the MS model lacks a second issue with certain gains from negotiation, and so it corresponds to a single-issue mediation problem. It is, however, easy to prove that there is no strategy-proof, efficient, and individually rational mediation mechanism in a single-issue mediation problem.

The mechanism design problem in MS concerns a bilateral trade between a buyer and a seller, who have private information about their valuations of a good. The mechanism has two components: the probability of trade, p, and the transfer, x, both of which are functions of the traders' reports. If no trade occurs, then $x = p = 0$ (the outside option), and so both traders receive zero utility. The utility functions are $U_b = v_b p - x$ for the buyer and $U_s = x - v_s p$ for the seller, where the valuations v_b, v_s are the traders' private information.

"Budget balancedness" is automatically satisfied in our setup, and so, "budget imbalance" is not the driving force for our possibility result. The buyer (seller) prefers lower (higher) transfers in MS and it is a priori uncertain whether a transfer leading to a mutually beneficial trade exists. Moreover, the quasi-linear utility functions in MS also satisfy the monotonicity and the quid pro quo assumptions. Despite all these similarities, the impossibility of MS is not at odds with our results because the MS model translates as a single-issue mediation problem in our setup, where the transfer is the issue with uncertain gains. Efficiency in MS implies that the probability of efficient trade is generically either 0 or 1, depending on whether or not the buyer's valuation is higher than the seller's valuation. This means that probability of trade cannot be considered as a second issue since we require the second issue to have at least as many alternatives as the main issue.

What is needed for a possibility is a new issue with a sufficient number of efficient alternatives as in the case of issue $\mathbf{Y}$ in our model. To provide an illustration of the above points, in the following example we offer a simple adaptation of the MS setup in our model and demonstrate how one can overcome the impossibility by adding an extra issue:

Example 4 (Possibility in the augmented MS framework): Suppose that the seller and the buyer now negotiate not only over the terms of trade but also over the division of a unit surplus, which is an issue independent from the terms of trade (the main issue). We refer to the latter as issue **Y**. The valuations of the good to the buyer and the seller are v_b and v_s, respectively. We assume that each negotiator knows her valuation and believes that the opponent's valuation is distributed over $[0, 1]$ with some probability distribution. The mediator privately solicits the traders' valuations and recommends a quadruple (p, x, y_s, y_b), where p denotes the probability of trade, x is the transfer, and y_s and y_b are respectively the seller's and the buyer's share of the unit surplus. The preferences of the two traders are as follows: $U_b = pv_b - x + u_b(y_b)$ and $U_s = x - pv_s + u_s(y_s)$. For simplicity, suppose that $u_b(y) = u_s(y) = y$ and each trader has only two types, $v_b, v_s \in \{0.2, 0.6\}$.

Efficiency implies that $p = 1$ if $v_b \geq v_s$, $p = 0$ if $v_s < v_b$, and $y_b + y_s = 1$. Individual rationality implies that the traders' utilities are nonnegative. In the absence of the second issue, **Y**, it is easy to show that there is no strategy-proof, efficient, and individually rational mechanism. However, the following mechanism is strategy-proof, efficient, and individually rational when the second issue **Y** is introduced:[50]

	$v_b = 0.6$		$v_b = 0.2$	
$v_s = 0.6$	$p = 1$	$y_s = 0.3$	*No*	$y_s = 0.5$
	$x = 0.6$	$y_b = 0.7$	*trade*	$y_b = 0.5$
$v_s = 0.2$	$p = 1$	$y_s = 0.5$	$p = 1$	$y_s = 0.7$
	$x = 0.4$	$y_b = 0.5$	$x = 0.2$	$y_b = 0.3$

6 Related Literature

Mediation is a highly interdisciplinary topic and our approach and analysis is novel relative to existing literature in several dimensions.

The law and economics literature on settlement negotiations under asymmetric information is extensive.[51] Our approach is fundamentally different from this literature both conceptually and methodologically. In a *settlement negotiation*, communications between parties revolve around evidence, rule of law, and witnesses; and when negotiations fail, trial is generally the next step. By contrast, we study what is referred as *facilitative mediation* in which the goal is not to determine who is right or who has a stronger case, but rather to explore mutually negotiable resolutions. In this type of mediation, mediator never invites parties to present their evidences or cases, nor allows parties to discuss their interpretation of the law. Its solution-oriented approach is what makes facilitative mediation the de facto dispute resolution method in e-commerce.

[50] The seller of type $v_s = 0.2$ has no incentive to mimic type $v_s = 0.6$. This is true because the seller's payoff under truth-telling (which is 0.7 regardless of the buyer's type) is higher than or equal to her deviation payoffs 0.7 (if the buyer is of type $v_b = 0.6$) and 0.5 (if the buyer is of type $v_b = 0.2$). Similarly, the seller of type $v_s = 0.6$ has no incentive to mimic type $v_s = 0.2$. Her payoff under truth-telling is either 0.3 (if the buyer is type $v_b = 0.6$) or 0.5 (if the buyer is type $v_b = 0.2$). However, her deviation payoffs are 0.3 regardless of the buyer's type. Symmetric arguments apply for the buyer.

[51] See, for example, Daughety and Reinganum (2017) and Wickelgren (2013) for two comprehensive accounts of this literature.

We are not aware of any paper that formally investigates the role of incentives in facilitative mediation.

From a modeling perspective, settlement negotiation models generally involve at least one party having private information about some aspects of the case. Parties' strengths determine the outcome of the trial and the value of the outside option for each side. As a direct implication of this modeling choice, settlement negotiation processes may reveal information about parties' strengths, which would mean updated beliefs and expectations about outside options. The mediator may also play a role in controlling the flow of information between the two sides. A model where parties can influence other parties' beliefs, and so preferences, creates a highly adversarial environment. However, the main idea behind facilitative mediation is to prevent the formation of such environments. In our model, parties' preferences (i.e., negotiable alternatives) do not change with the opponent's private information. Settlement negotiations are generally modeled through the lens of traditional bargaining models (e.g., Nash 1953, Rubinstein 1982, and Myerson and Satterthwaite 1983). However, existing bargaining models offer limited insights into the practicality of offering compromises in multi-dimensional deals; a wisdom often voiced by experts in the field (Fish and Ury 1983 and Malhotra Bazerman 2008).

A central question in bargaining under incomplete information is whether private information prevents the bargainers from reaping all possible gains from trade. The pioneering work of Myerson and Satterthwaite (1983) [MS] gave a negative answer: in a model with transferable utility there is no ex post efficient, individually rational, Bayesian incentive compatible, and budget balanced mechanism under uncertain gains from trade.[52] In this literature only a limited number of papers study the topic of mediation with outside options.[53] However, their focus is also on settlement negotiations. Our modeling of outside options is more in line with how issues are addressed in political bargaining; see, e.g., Chen and Eraslan (2014, 2017).

Obtaining a possibility result in our model hinges crucially on the availability of a supplementary issue. Linking multiple issues to overcome welfare and incentive constraints has been a useful tool in many economic applications such as bundling of goods by a monopolist (e.g., McAfee et al. 1989), agency problems (e.g., Maskin and Tirole 1990), and logrolling in voting (e.g., Wilson 1969). A common insight in these approaches is based on applying a law of large numbers theorem to ensure that truth telling incentives are restored in a sufficiently large market. In this vein, Jackson and Sonnenschein (2007) show that by linking different issues in many situations, including the bilateral bargaining setting of MS, it is possible to achieve outcomes that are approximately efficient in an approximately incentive compatible way as the number of issues goes to infinity. In contrast with these approaches, we establish efficiency in

[52] Recent empirical work on used car sales by Larsen (2021) confirms the predictions in MS. More than half of failed negotiations in the industry involved situations where gains from trade actually existed. On the other hand, the MS impossibility crucially depends on types being independent. Subsequently, it was shown that efficient trade may be possible when types are correlated; see e.g., Gresik (1991) and McAfee and Reny (1992).

[53] Bester and Warneryd (2006) show that asymmetric information about relative strengths as an outside option in a conflict may render agreement impossible even if there is no uncertainty about the agreement being efficient. Hörner et al. (2015) compare the optimal mechanisms with two types of negotiators under arbitration, mediation, and unmediated communication. Compte and Jehiel (2009) consider a bargaining problem where outside options are private but correlated, and parties have a veto right. They show that inefficiencies are inevitable whatever the exact form of correlation, which resonates with the negative result in a model of single-issue mediation.

dominant strategies with only two issues in an application where the number of potential issues is inherently limited.

Departing from the mechanism design literature and in similar spirit to us, Jackson et al. (2021) argue that "Existing bargaining models shed no light on [the] perceived wisdom [of practitioners] that offering multiple deals and searching for the right one is central to negotiations." They also emphasize that mechanisms that are based on the utility functions or beliefs of the agents can be viewed as impractical (see also Wilson 1987 and Satterthwaite, Williams and Zachariadis 2014). The practical formulation of our logrolling mechanisms as shortlisting mechanisms with a menu of offers (Section 3.4) directly addresses their former critique, while our ordinal approach together with dominant strategy implementation addresses the latter.[54]

With some caveats, a dispute resolution problem can also be interpreted as a type of fair division problem involving indivisible items. Logrolling mechanisms allow one negotiator to effectively reduce the set of possible outcomes to a shortlist, from which the other negotiator makes her favorite selection. In that sense, logrolling mechanisms are reminiscent of the well-known biblical rule of divide-and-choose, which has been extensively studied in fair cake-cutting problems. Two advantages of a logrolling mechanism relative to divide-and-choose is that it is strategy-proof (whenever preferences satisfy quid pro quo) and its outcome is independent of the ordering of the negotiators. More generally, the fair division literature almost exclusively focuses on fairness and efficiency issues due to inherent incompatibilities with strategy-proofness similar to those in the multi-unit assignment context; see Brams and Taylor (1996) for an excellent overview.

Assignment problems have proved useful in achieving strategy-proofness and efficiency via non-dictatorial mechanisms in a number of applications. In this context, ordinal mechanisms are well known to achieve better incentive properties than their cardinal contenders.[55] In early work, Zhou (1990) showed that no cardinal mechanism is strategy-proof, efficient, and symmetric. By contrast, ordinal mechanisms such as the random priority, are well known to attain the three properties. In two-sided matching problems, however, a stable mechanism can be strategy-proof only for one side of the market (see e.g., Roth and Sotomayor 1990).[56]

In assignment/matching problems, an agent's outside option is private consumption whereas in our model it creates an externality on the other negotiator, e.g., whenever either negotiator chooses to exercise her outside option by vetoing the proposal, the other negotiator is automatically compelled to also exercise her outside option. When outside options do not exist and the issues are discrete, our setting roughly resembles a type of multi-unit assignment problem

[54]Jackson et al. (2021) revisit the setting of Jackson and Sonnenschein (2007) and investigate whether approximately efficient outcomes can be obtained in the absence of a mechanism designer or any constraints on agents' preferences or the offers they can make.

[55]For example, the most prominent cardinal mechanism in the context of unit-assignment problems (possibly allowing for stochastic assignments), the competitive equilibrium from equal incomes solution (Hylland and Zeckhauser 1979), is not strategy-proof. This difficulty of achieving strategy-proofness is generally attributed to the tension with efficiency since cardinal mechanisms achieve stronger welfare properties (e.g., maximization of utilitarian welfare) than ordinal mechanisms.

[56]Similar to the literature on linking decisions discussed below, a common method of circumventing these impossibilities is to resort to large market arguments by allowing for the number of participants and resources to grow. Such methods are obviously inapplicable in the context of mediation.

(e.g., course allocation) in which only certain assignments are feasible.[57] As discussed in Section 1.1, some of the existing point allocation-based ODR mechanisms are akin to course-bidding mechanisms that have been discussed in this literature. Nonetheless, the multi-unit assignment setting provides little reason to remain optimistic for positive results. The literature contains a series of papers that show impossibility results. The main result of this literature is that the only strategy-proof and efficient mechanisms are serial dictatorships; e.g., see Pápai (2001), Klaus and Miyagawa (2002), and Ehlers and Klaus (2003).[58] Clearly, dictatorship mechanisms have little appeal in a dispute resolution situation.[59]

In voting, special domains allow for positive results. The celebrated median voter theorem states that the majority-rule voting system that selects the Condorcet winner (i.e., the outcome most preferred by the median voter) is strategy-proof; see Moulin (1980) for a classic generalization of this result. Our constrained shortlisting mechanism can be viewed as similar to a Condorcet winner in the sense that it recommends the median logrolling bundle when the median is mutually negotiable for both negotiators and the closest logrolling bundle to it when it is not. Nevertheless, this connection is superficial as our model has no direct comparison to a voting model.[60]

A novelty of our approach that distinguishes it from the literature on fair division, matching, and voting is that we do not ask agents to report their preferences. Instead, we maintain the view that a dispute is solvable so long as the underlying preferences allow for it. This not only frees us from further complications related to the choice of a suitable preference reporting language, but is also consistent with many of the ODR practices as well as the recommendation systems used in e-commerce.

Finally, with the hope of arriving at possibility results, there is a tradition of searching for strategy-proof mechanisms in restricted economic environments that make it possible to escape the famous Arrow-Gibbard-Satterthwaite impossibility results. Well-known examples include the VCG mechanisms (Vickrey 1961, Groves 1973, and Clarke 1971) for public goods and private assignment with transfers; the uniform rule (Sprumont 1991) for the distribution of a divisible private good under single-peaked preferences; generalized median-voters (Moulin 1980);

[57]Suppose there are two agents, each of whom needs to be assigned two objects, one from each of two sets A and B, where an alternative in issue $\mathbf{X}$ (respectively $\mathbf{Y}$) represents a specific pair of objects from set A (respectively B) that must be assigned simultaneously. Suppose, for example, set A contains three objects in the order of decreasing desirability, a, b, and c, where a and c are in unit supply and b has two copies. Then issue $\mathbf{X}$ can be viewed as consisting of the following object pairings $\mathbf{X} = \{(a,c),(b,b),(c,a)\}$. That is, if one agent gets a, the other must get c, and b cannot be assigned together with any other object.

[58]Results continue to be negative even with stochastic mechanisms (Kojima, 2009). In the course allocation context, two notable contributions that identify non-dictatorship mechanisms are Sönmez and Ünver (2010) and Budish (2011). The former paper argues for eliciting bids from students together with ordinal preferences over courses and then using a Gale-Shapley mechanism where bids are interpreted as course priorities. The mechanism is strategy-proof only if the bids are treated as exogenously given. The latter paper proposes an approximately efficient mechanism that is strategy-proof in a large market.

[59]Worse still, dictatorships violate individual rationality in our model, i.e., such recommendations will be vetoed in equilibrium. A constrained dictatorship where one negotiator maximizes her welfare among the set of mutually negotiable outcomes would satisfy individual rationality, but such a mechanism is easily manipulable.

[60]An alternative comparison could be with multi-dimensional voting models. A main conclusion in such models is that strategy-proofness effectively requires each dimension to be treated separately in the sense that each dimension should admit its own generalized median voter schemes. Our strategy-proofness result, by contrast, depends critically on having more than one dimension and leveraging the substitution between the two issues.

proportional-budget exchange rules (Barberà and Jackson 1995) that allow for trading from a finite number of prespecified proportions (budget sets); deferred acceptance (Gale and Shapley 1962) and top trading cycles (Shapley and Scarf 1974, Abdülkadiroglu and Sönmez 2003) and hierarchical exchange and brokerage (Pápai 2001, Pycia and Ünver 2017). We also add to this literature by introducing and characterizing an entirely new class of strategy-proof and efficient mechanisms.

7 Conclusion

Mediation is a preferred alternative dispute resolution method thanks to the cost-effectiveness, speed, and convenience it affords to all parties involved. The need for structured and rigorous mediation protocols in practice has often been stressed by researchers and practitioners alike. Online dispute resolution platforms are often based on automation and rely on mechanized negotiation protocols. However, existing dispute resolution protocols do not account for the incentives faced by disputants. Taking a foundational mechanism design approach to this problem, we sought systematic mechanisms for delivering consistent, transparent, and objective recommendations while giving strong incentives to the disputants to be truthful. We considered mechanisms that have a simple preference reporting language; negotiators only report their bargaining ranges (i.e., least negotiable alternatives) in the main issue. It turns out that complementing the main issue with a second one—a piece of advice often raised by pioneers in the field—is key to achieving strategy-proof, efficient, and individually rational mechanisms. Any such mechanism belongs to the family of logrolling mechanisms, which require that the mediator's recommendation must always be a logrolling bundle (a bundle that complements a more preferred alternative in one issue with a less preferred alternative from the other) when a mutual agreement is feasible. A sufficient and necessary condition for strategy-proofness is the quid pro quo property of preferences that necessitates the alternatives in the second issue to be interesting enough to allow for preference reversals. The constrained shortlisting mechanism is the central member within the characterized class and makes recommendations as close to the median logrolling bundle as possible.

Our approach can also be viewed as a novel attempt to marry the two distinct strands of literature on bargaining and assignment. Although the design of dominant strategy incentive compatible facilitative mediation protocols has not been previously considered in the former, this literature emphasizes the tensions due to private information and outside options in mechanism design with transferable utility. The latter literature offers blueprints for designing robust protocols in assignment problems that often arise in practice. The multiple-assignment nature of the problem at hand in our study, however, is less than encouraging in light of the abundance of negative results in that literature. Our analysis confirms these challenges in that possibility results in our framework are also elusive unless the two issues are treated asymmetrically. We argued that ordinal mechanisms coupled with strategy-proofness can help obtain detail-free and genuinely simple protocols for mediating disputes. Notwithstanding our emphasis on ordinality, the framework developed in this paper can accommodate both transferable and nontransferable utility settings since we do not directly elicit preferences.

While it would be premature to conclude that logrolling mechanisms are ready-to-use protocols for immediate practical applications, our theoretical analysis may help shed light on the fundamental forces at work when efficiency is sought together with robust incentives. An interesting open question is how to incorporate full preference elicitation from negotiators into the mechanism design problem. Further research is needed on this front since allowing negotiator types to also include preferences would readily give rise to a more sophisticated preference reporting language than ours as well as new incentive and welfare considerations. Although we leave this direction for future investigation, we contend that the class of mechanisms characterized here would constitute an ideal starting point for developing more sophisticated dispute resolution protocols.

8 APPENDIX

We start with establishing a series of preliminary results. For any mechanism f and type profile $t = (x^1_\ell, x^2_j) \in \mathbf{T}$, we let $f(t) = (f^{\mathbf{X}}_t, f^{\mathbf{Y}}_t) = (f^{\mathbf{X}}_{\ell,j}, f^{\mathbf{Y}}_{\ell,j}) = f_{\ell,j}$.

Lemma 0. *Rationalizable utility functions satisfy* ***type invariance****: Namely, for all $i \in \mathbf{I}$, all $U_i \in \mathcal{U}^R_i$, all $t_i, t'_i \in \mathbf{T}_i$, all $x, x' \in \mathbf{N}(t_i) \cap \mathbf{N}(t'_i)$, and all $y, y' \in \mathbf{Y}$, we have*

$$U_i(x, y, t_i) \geq U_i(x', y', t_i) \iff U_i(x, y, t'_i) \geq U_i(x', y', t'_i).$$

Furthermore, rationalizable utility functions are monotonic over the set of acceptable bundles.

Proof. Take an arbitrary $i \in \mathbf{I}$, $U_i \in \mathcal{U}^R_i$, $t_i, t'_i \in \mathbf{T}_i$, $x, x' \in \mathbf{N}(t_i) \cap \mathbf{N}(t'_i)$, and $y, y' \in \mathbf{Y}$. Because U_i is a rationalizable utility function, there must exits a $u_i \in \mathcal{U}_i$ such that $U_i = u^R_i$. Namely, $U_i(x, y, t_i) = u_i(x, y) = U_i(x, y, t'_i)$ and $U_i(x', y', t_i) = u_i(x', y') = U_i(x', y', t'_i)$ because $x, x' \in \mathbf{N}(t_i) \cap \mathbf{N}(t'_i)$. This completes the first part of the proof. For the second part, recall that a rationalizable utility function $U_1 \in \mathcal{U}^R_1$ (respectively, $U_2 \in \mathcal{U}^R_2$) is identical to some decreasing (respectively, increasing) function $u_1 \in \mathcal{U}_1$ (respectively, $u_2 \in \mathcal{U}_2$) over all acceptable bundles. Therefore, rationalizable utility functions are monotonic over the set of acceptable bundles. □

Lemma 1. *If mechanism f is efficient and individually rational over some non-empty set $\mathcal{D} \subseteq \mathcal{U}$, then for any $t = (x^1_\ell, x^2_j) \in \mathbf{T}$ with $j \leq \ell$ we have $f^{\mathbf{X}}_t \in [x_j, x_\ell]$, where $[x_j, x_\ell] = \mathbf{N}(x^1_\ell) \cap \mathbf{N}(x^2_j)$ denotes the set of all mutually negotiable alternatives at type profile t.*

Proof. Take any $t = (x^1_\ell, x^2_j) \in \mathbf{T}$ with $j \leq \ell$, and so $\mathbf{N}(x^1_\ell) = \{x_1, \ldots, x_j, \ldots, x_\ell\}$ and $\mathbf{N}(x^2_j) = \{x_j, \ldots, x_\ell, \ldots, x_m\}$. Therefore, $\mathbf{N}(x^1_\ell) \cap \mathbf{N}(x^2_j) = [x_j, x_\ell]$. Because f is efficient over $\mathcal{D}$ and negotiators' utility functions $U_1 \in \mathcal{D}^R_1$ and $U_2 \in \mathcal{D}^R_2$ satisfy deal-breaker property, $f(t) \neq \phi$. Moreover, we must have $f^{\mathbf{X}}_t \notin \{x_1, \ldots, x_{j-1}, x_{\ell+1}, \ldots, x_m\}$ because f is individually rational over $\mathcal{D}$, alternatives in $\{x_{\ell+1}, \ldots, x_m\}$ are non-negotiable dealbreaker for type x^1_ℓ of Negotiator 1, alternatives in $\{x_1, \ldots, x_{j-1}\}$ are non-negotiable dealbreaker for type x^2_j of Negotiator 2, and every $(U_1, U_2) \in \mathcal{D}^R$ satisfy deal-breaker property. Hence, it must be that $f^{\mathbf{X}}_t \in [x_j, x_\ell]$. □

Lemma 2 (WARP). *Assume that mechanism f is strategy-proof, efficient, and individually rational over some non-empty set $\mathcal{D} \subseteq \mathcal{U}$. For any $t, t' \in \mathbf{T}$ and distinct alternatives $x, x' \in \mathbf{X}$, satisfying $x, x' \in \mathbf{N}(t_i) \cap \mathbf{N}(t'_i)$ for $i = 1, 2$, if $f^{\mathbf{X}}_t = x$, then $f^{\mathbf{X}}_{t'} \neq x'$.*

Proof. Assume that f is strategy-proof, efficient, and individually rational over $\varnothing \neq \mathcal{D} \subseteq \mathcal{U}$, and suppose for a contradiction that there are $t = (t_1, t_2) = (x^1_\ell, x^2_j) \in \mathbf{T}$, $t' = (t'_1, t'_2) = (x^1_{\ell'}, x^2_{j'}) \in \mathbf{T}$, and distinct alternatives $x_k, x_{k'} \in \mathbf{X}$, satisfying $x_k, x_{k'} \in \mathbf{N}(t_i) \cap \mathbf{N}(t'_i)$ for $i = 1, 2$, such that $f^{\mathbf{X}}_t = x_k$ and $f^{\mathbf{X}}_{t'} = x_{k'}$. Suppose, without loss of generality, that $\ell < \ell'$ (if $\ell = \ell'$, then just skip the steps involving bundle b and Negotiator 1 below). Consider the less accepting type of Negotiator 1 (i.e., x^1_ℓ) and type profile $(x^1_\ell, x^2_{j'})$. It must be the case that $j' \leq \ell$. To prove this last claim, suppose for a contradiction that $j' > \ell$. Because $x_k \in \mathbf{N}(x^1_\ell)$ and $x_k \in \mathbf{N}(x^2_{j'})$ we have $k \leq \ell$ and $k \geq j'$. These two inequalities imply $j' \leq k \leq \ell$, contradicting with $j' > \ell$.

Therefore, Lemma 1 requires that $f(t_1, t'_2) = b$ for some $b \in \mathbf{B} \setminus \{\phi\}$. Note that all three bundles (i.e., b, $f(t)$ and $f(t')$) are acceptable by two types of Negotiator 1. Bundles $f(t)$ and $f(t')$ are acceptable by both because $x_k, x_{k'} \in \mathbf{N}(t_1) \cap \mathbf{N}(t'_1)$. Bundle b is also acceptable by both types of Negotiator 1 because f is individually rational and suggests b at type profile (t_1, t'_2), meaning b is acceptable for type t_1, and t'_1 is more accepting than t_1.

Strategy-proofness requires $U_1(b, t_1) \geq U_1(f(t'), t_1)$ for all $U_1 \in \mathcal{D}^R_1$, and type-invariance of U_1 implies $U_1(b, t'_1) \geq U_1(f(t'), t'_1)$ because both b and $f(t')$ are acceptable by types t_1 and t'_1. Strategy-proofness also requires $U_1(f(t'), t'_1) \geq U_1(b, t'_1)$ for all $U_1 \in \mathcal{D}^R_1$. The last two inequalities imply $b = f(t')$ because U_1's are strict. Moreover, strategy-proofness requires $U_2(f(t), t_2) \geq U_2(f(t'), t_2)$ for all $U_2 \in \mathcal{D}^R_2$ because $f(t_1, t'_2) = b$ and $b = f(t')$, and type-invariance of U_2's implies $U_2(f(t), t'_2) \geq U_2(f(t'), t'_2)$ because $f(t)$ and $f(t')$ are acceptable by both t_2 and t'_2. (This last claim is true because $f^{\mathbf{X}}_{t_1, t'_2} = x_{k'}$ and $x_k, x_{k'} \in \mathbf{N}(t_2) \cap \mathbf{N}(t'_2)$). Strategy-proofness also requires $U_2(f(t'), t'_2) \geq U_2(f(t), t'_2)$ for all $U_2 \in \mathcal{D}^R_2$. Because U_2's are strict, the last two inequalities imply $f(t) = f(t')$, contradicting that x_k and $x_{k'}$ are distinct alternatives. □

Lemma 3. *If mechanism f is strategy-proof, efficient, and individually rational over some non-empty set $\mathcal{D} \subseteq \mathcal{U}$, then there exists an injective and decreasing function $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ such that $f_{k,k} = (x_k, \mathbf{y}(x_k))$ for every $k \in \mathcal{I}$.*

Proof. Assume that mechanism f is strategy-proof, efficient, and individually rational over $\varnothing \neq \mathcal{D} \subseteq \mathcal{U}$. Lemma 1 implies $f^{\mathbf{X}}_{k,k} = x_k$ for any $k \in \mathcal{I}$ and $f^{\mathbf{X}}_{k+1,k} \in \{x_k, x_{k+1}\}$ whenever $k \neq m$. Furthermore, it must be that $f_{k+1,k} \in \{f_{k,k}, f_{k+1,k+1}\}$. To prove this suppose, for a contradiction, that $f_{k+1,k} \notin \{f_{k,k}, f_{k+1,k+1}\}$. If $f^{\mathbf{X}}_{k+1,k} = x_k$, which is equal to $f^{\mathbf{X}}_{k,k}$, then strategy-proofness and monotonicity of $U_1 \in \mathcal{D}^R_1$ require $f^{\mathbf{Y}}_{k+1,k} = f^{\mathbf{Y}}_{k,k}$, namely $f_{k+1,k} = f_{k,k}$. On the other hand, if $f^{\mathbf{X}}_{k+1,k} = x_{k+1}$, which is equal to $f^{\mathbf{X}}_{k+1,k+1}$, then strategy-proofness and monotonicity of $U_2 \in \mathcal{D}^R_2$ require $f^{\mathbf{Y}}_{k+1,k} = f^{\mathbf{Y}}_{k+1,k+1}$, namely $f_{k+1,k} = f_{k+1,k+1}$, leading to the desired contradiction.

Next, we set $\mathbf{y}(x_k) = f^{\mathbf{Y}}_{k,k}$ for $k \in \mathcal{I}$. To prove that $\mathbf{y}$ is injective and decreasing, we prove the following: For any $k \in \mathcal{I} \backslash \{m\}$, if $\mathbf{y}(x_k) = y_\ell$ and $\mathbf{y}(x_{k+1}) = y_{\ell'}$, then $\ell' < \ell$. Suppose for a contradiction that there is some k such that $\mathbf{y}(x_k) = y_\ell$, $\mathbf{y}(x_{k+1}) = y_{\ell'}$ and $\ell' \geq \ell$. Recall that $f_{k+1,k} \in \{f_{k,k}, f_{k+1,k+1}\}$. If $f_{k+1,k} = f_{k+1,k+1}$, then strategy-proofness require that $U_1(f_{k+1,k+1}, t_1) \geq U_1(f_{k,k}, t_1)$ for all $U_1 \in \mathcal{D}^R_1$, where $t_1 = x^1_{k+1}$. Because $f_{k+1,k+1} = (x_{k+1}, \mathbf{y}(x_{k+1})) = (x_{k+1}, y_{\ell'})$

and $f_{k,k} = (x_k, \mathbf{y}(x_k)) = (x_k, y_\ell)$, the last inequality implies $U_1(x_{k+1}, y_{\ell'}, t_1) \geq U_1(x_k, y_\ell, t_1)$, contradicting that U_1's are decreasing and $\ell' \geq \ell$. However, if $f_{k+1,k} = f_{k,k}$, then strategy-proofness require $U_2(f_{k,k}, t_2) \geq U_2(f_{k+1,k+1}, t_2)$ for all $U_2 \in \mathcal{D}_2^R$. This inequality implies that $U_2(x_k, y_\ell, t_2) \geq U_2(x_{k+1}, y_{\ell'}, t_2)$, contradicting that U_2 is increasing and $\ell' \geq \ell$. Hence, it must be that $\ell' < \ell$, so $\mathbf{y}$ is injective and decreasing. $\square$

Lemma 4. *If mechanism f is strategy-proof, efficient, and individually rational over some non-empty set $\mathcal{D} \subseteq \mathcal{U}$, then for any $t = (x_\ell^1, x_j^2) \in \mathbf{T}$, we have*

$$f(t) = f(x_\ell^1, x_j^2) = \begin{cases} \left(x^*_{[x_j,x_\ell]}, \mathbf{y}\left(x^*_{[x_j,x_\ell]}\right)\right) & \text{if } j \leq \ell \\ \phi & \text{otherwise;} \end{cases}$$

where $x^*_{[x_j,x_\ell]} \in [x_j, x_\ell]$ and $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ is an injective and decreasing function.

Proof. Assume that mechanism f is strategy-proof, efficient, and individually rational over $\varnothing \neq \mathcal{D} \subseteq \mathcal{U}$. Lemma 3 ensures that there is an injective and decreasing function $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ such that $f_{k,k} = (x_k, \mathbf{y}(x_k))$ for every $k \in \mathcal{I}$. Let $\mathbf{B}^\mathbf{y} = \{(x_k, \mathbf{y}(x_k)) | x_k \in \mathbf{X}\}$ be the set of all bundles on the main (first) diagonal. Take any $t = (x_\ell^1, x_j^2) \in \mathbf{T}$ where $\ell, j \in \mathcal{I}$. If $\ell < j$, then efficiency and individual rationality of f and the deal-breaker property of all $(U_1, U_2) \in \mathcal{D}^R$ require $f(t) = \phi$.

Therefore, we assume $j < \ell$ for the rest of the proof, and prove that $f_{\ell,j} = (x^*_{[x_j,x_\ell]}, \mathbf{y}(x^*_{[x_j,x_\ell]})) \in \mathbf{B}^\mathbf{y}$ for some $x^*_{[x_j,x_\ell]} \in [x_j, x_\ell]$. To show this we suppose, for a contradiction, that there exists such j and ℓ such that either $f_{\ell,j} \notin \mathbf{B}^\mathbf{y}$ or $x^*_{[x_j,x_\ell]} \notin [x_j, x_\ell]$ holds. Lemma 1 requires that $f^\mathbf{X}_{\ell,j} = x^*_{[x_j,x_\ell]} \in [x_j, x_\ell]$. Therefore, it must be that $x^*_{[x_j,x_\ell]} = x_k$ for some $j \leq k \leq \ell$ and $f_{\ell,j} \notin \mathbf{B}^\mathbf{y}$. Lemma 2 requires $f^\mathbf{X}_{s,r} = x_k$ for all $k \leq s \leq \ell$ and $j \leq r \leq k$ because $f^\mathbf{X}_{k,k} = x_k$ (by Lemma 3), $x_k \in \mathbf{N}(x_s^1) \cap \mathbf{N}(x_r^2)$ for all s and r satisfying $k \leq s$ and $r \leq k$, and $x^*_{[x_j,x_\ell]} = x_k$. Consider now the type profile (x_k^1, x_j^2). Strategy-proofness of f and monotonicity of any $U_2 \in \mathcal{D}_2^R$ require that $f_{k,j} = f_{k,k}$ because $f^\mathbf{X}_{k,j} = x_k = f^\mathbf{X}_{k,k}$. Similarly, strategy-proofness of f and monotonicity of any $U_1 \in \mathcal{D}_1^R$ require $f_{k,j} = f_{\ell,j}$ because $f^\mathbf{X}_{\ell,j} = x_k = f^\mathbf{X}_{k,j}$. Thus, it must be that $f_{\ell,j} = f_{k,k} = (x_k, \mathbf{y}(x_k))$, contradicting that $f_{\ell,j} \notin \mathbf{B}^\mathbf{y}$. Hence, we have $f_{\ell,j} = (x^*_{[x_j,x_\ell]}, \mathbf{y}(x^*_{[x_j,x_\ell]})) \in \mathbf{B}^\mathbf{y}$ for some $x^*_{[x_j,x_\ell]} \in [x_j, x_\ell]$. $\square$

Lemma 5. *If mechanism f is strategy-proof, efficient, and individually rational over some non-empty set $\mathcal{D} \subseteq \mathcal{U}$, then there exists a partial order $\trianglerighteq$ on $\mathbf{X}$ such that for any $t = (x_\ell^1, x_j^2) \in \mathbf{T}$ with $j \leq \ell$, $f^\mathbf{X}_{\ell,j} \trianglerighteq x$ for all $x \in [x_j, x_\ell]$.*

Proof. Assume that mechanism f is strategy-proof, efficient, and individually rational over $\varnothing \neq \mathcal{D} \subseteq \mathcal{U}$. We construct $\trianglerighteq$ as follows: For any $x_k, x_{k'} \in \mathbf{X}$

$$x_k \;\trianglerighteq\; x_{k'} \iff f^\mathbf{X}_{\max\{k,k'\},\min\{k,k'\}} = x_k.$$

By construction, $\trianglerighteq$ is antisymmetric and reflexive. To prove that $\trianglerighteq$ is a partial order on $\mathbf{X}$ we need to show that it is transitive; namely for any triple $x, x', x'' \in \mathbf{X}$ where $x \trianglerighteq x'$ and $x' \trianglerighteq x''$, we have $\neg x'' \trianglerighteq x$. Suppose for a contradiction that there exists three distinct $x_k, x_{k'}, x_{k''} \in \mathbf{X}$ such that $x_k \trianglerighteq x_{k'}$, $x_{k'} \trianglerighteq x_{k''}$, and $x_{k''} \trianglerighteq x_k$. Suppose, without loss of generality, that $k < k' < k''$.

By construction of $\trianglerighteq$, it must be that $f^{\mathbf{X}}_{k'',k'} = x_{k'}$ and $f^{\mathbf{X}}_{k'',k} = x_{k''}$. Moreover, Lemma 4 implies $f^{\mathbf{X}}_{k'',k'} \in [x_{k'}, x_{k''}]$ and $f^{\mathbf{X}}_{k'',k} \in [x_k, x_{k''}]$: Namely, f suggests $x_{k'}$ (i.e., $f^{\mathbf{X}}_{k'',k'} = x_{k'}$) while both $x_{k'}$ and $x_{k''}$ are mutually negotiable (i.e., $x_{k'}, x_{k''} \in [x_{k'}, x_{k''}]$) and suggests $x_{k''}$ (i.e., $f^{\mathbf{X}}_{k'',k} = x_{k''}$) while these two alternatives are still mutually negotiable ($x_{k'}, x_{k''} \in [x_k, x_{k''}]$ because $k < k' < k''$), contradicting (by Lemma 2) that f is strategy-proof, efficient, and individually rational over $\mathcal{D}$. Thus, $\trianglerighteq$ is transitive.

Now take any $t = (x^1_\ell, x^2_j) \in \mathbf{T}$ with $j \leq \ell$ and set $f^{\mathbf{X}}_{\ell,j} = x_s$ for some $j \leq s \leq \ell$. To show $f^{\mathbf{X}}_{\ell,j} \trianglerighteq x$ for all $x \in [x_j, x_\ell]$, suppose for a contradiction that there exists $x_{s'} \in [x_j, x_\ell]$ such that $\neg x_s \trianglerighteq x_{s'}$. Suppose, without loss of generality, $s' < s$. Because $\neg x_s \trianglerighteq x_{s'}$, construction of $\trianglerighteq$ requires that $f^{\mathbf{X}}_{s,s'} \neq x_s$. Let $f^{\mathbf{X}}_{s,s'} = x_{s''}$ for some $x_{s''} \in [x_{s'}, x_s]$. With all the given information, consider the following two profiles; $t = (x^1_\ell, x^2_j)$ and $t' = (x^1_s, x^2_{s'})$. We have $x_{s''}, x_s \in \mathbf{N}(t_i) \cap \mathbf{N}(t'_i)$ for $i = 1, 2$, $f^{\mathbf{X}}_t = x_s$ and $f^{\mathbf{X}}_{t'} = x_{s''}$, contradicting (by Lemma 2) that f is strategy-proof, efficient and individually rational over $\mathcal{D}$. Hence, it must be that $f^{\mathbf{X}}_{\ell,j} = \underset{[x_j, x_\ell]}{\boldsymbol{max}} \trianglerighteq$. □

Proof of Theorem 1: Proof immediately follows from Lemmata 2-4. ■

Lemma 6. *If mechanism $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism for some partial order $\succeq_\mathbf{y}$ on $\mathbf{X}$, then for any distinct logrolling bundles $b, b' \in \mathbf{B}^\mathbf{y}$, where $b = (x_k, \mathbf{y}(x_k))$ and $b' = (x_{k'}, \mathbf{y}(x_{k'}))$, and $\ell, \ell', j, j' \in \mathcal{I}$, where $j \leq \ell$, $j' \leq \ell'$, $f^{\succeq_\mathbf{y}}_{\ell,j} = b$, and $f^{\succeq_\mathbf{y}}_{\ell',j'} = b'$, we have the following:*

(*i*) $b \in V(b) \equiv \left\{ f^{\succeq_\mathbf{y}}_{r,s} | r, s \in \mathcal{I} \text{ and } s \leq k \leq r \right\}$.

(*ii*) $\{b, b'\} \nsubseteq V(b) \cap V(b')$.

(*iii*) *If $\ell' < \ell$ and $j' = j$, then $k' \leq k$.*

(*iv*) *If $\ell' = \ell$ and $j' < j$, then $k' \leq k$.*

Proof. Assume that $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism, namely for any $r, s \in \mathcal{I}$ with $s \leq r$, $f^{\succeq_\mathbf{y}}_{r,s} = \left(x^*_{[x_s,x_r]}, \mathbf{y}(x^*_{[x_s,x_r]})\right)$ where $x^*_{[x_s,x_r]} \succeq_\mathbf{y} x$ for all $x \in [x_s, x_r]$. Take any two bundles $b, b' \in \mathbf{B}^\mathbf{y}$, satisfying $b = (x_k, \mathbf{y}(x_k))$, $b' = (x_{k'}, \mathbf{y}(x_{k'}))$, and $x_k \neq x_{k'}$, and any indices $\ell, \ell', j, j' \in \mathcal{I}$, satisfying $j \leq \ell$, $j' \leq \ell'$, $f^{\succeq_\mathbf{y}}_{\ell,j} = b$, and $f^{\succeq_\mathbf{y}}_{\ell',j'} = b'$.

To prove claim of (i), suppose for a contradiction that $b \notin V(b)$. Namely, either $j \leq \ell < k$ or $\ell \geq j > k$ holds. In either case we have $x_k \notin [x_j, x_\ell]$, contradicting that $f^{\succeq_\mathbf{y}}_{\ell,j}$ is a logrolling mechanism. To prove claim of (ii), suppose for a contradiction that $\{b, b'\} \subseteq V(b) \cap V(b')$; namely, $j \leq k, k' \leq \ell$ and $j' \leq k, k' \leq \ell'$. These two inequalities imply $x_k, x_{k'} \in [x_j, x_\ell] \cap [x_{j'}, x_{\ell'}]$. Because $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism and $f^{\succeq_\mathbf{y}}_{\ell,j} = b$, it must be that $x_k = x^*_{[x_j,x_\ell]}$, and so $x_k \succeq_\mathbf{y} x_{k'}$. Similarly, because $f^{\succeq_\mathbf{y}}_{\ell',j'} = b'$, it must be that $x_{k'} \succeq_\mathbf{y} x_k$, contradicting that $\succeq_\mathbf{y}$ is antisymmetric and $x_k \neq x_{k'}$. To prove claim of (iii), assume that $\ell' < \ell$ and $j' = j$, and suppose for a contradiction that $k < k'$. Because $f^{\succeq_\mathbf{y}}_{\ell,j} = b$ and $f^{\succeq_\mathbf{y}}_{\ell',j'} = b'$, claim of ($i$) implies $j \leq k \leq \ell$ and $j' \leq k' \leq \ell'$. Because $\ell' < \ell$ and $k < k'$, it must be that $j' = j \leq k < k' \leq \ell' < \ell$; namely $\{b, b'\} \subseteq V(b) \cap V(b')$, contradicting the claim of (ii). Symmetric arguments suffice to prove claim of (iv). □

Lemma 7. *If the non-empty set $\mathcal{D} \subseteq \mathcal{U}$ satisfies quid pro quo and $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism, where the partial order $\succeq_\mathbf{y}$ is in $\Pi_\mathcal{D}$, then $f^{\succeq_\mathbf{y}}$ is efficient over $\mathcal{D}$.*

Proof. Assume that $\mathcal{D}$ satisfies quid pro quo and $f^{\succeq_{\mathbf{y}}}$ is a logrolling mechanism, where $\succeq_{\mathbf{y}} \in \Pi_{\mathcal{D}}$. Take any type profile $t = (x^1_\ell, x^2_j) \in \mathbf{T}$. If $j > \ell$, then $f^{\succeq_{\mathbf{y}}}(t) = \phi$. By deal-breaker property, no bundle in $\mathbf{B}\backslash\{\phi\}$ would make one negotiator better off without hurting the other. Suppose, for the rest of the proof, that $j \leq \ell$. Because $f^{\succeq_{\mathbf{y}}}$ is a logrolling mechanism, it must be that $f^{\succeq_{\mathbf{y}}}(t) = (x_k, \mathbf{y}(x_k)) \in \mathbf{B}^{\mathbf{y}}$ for some $k \in [x_j, x_\ell]$. Next we show that any other bundle would make at least one negotiator worse off, and thus conclude that $f^{\succeq_{\mathbf{y}}}$ is efficient. We prove this claim in two steps:

For step 1, we prove that any bundle in $\mathbf{B}^{\mathbf{y}}\backslash\{(x_k, \mathbf{y}(x_k))\}$ would make one of the negotiators worse off. To show this we suppose, for a contradiction, that there exists a bundle $(x_{k'}, \mathbf{y}(x_{k'})) \in \mathbf{B}^{\mathbf{y}}\backslash\{(x_k, \mathbf{y}(x_k))\}$, which is acceptable by both negotiators at type profile t, and that $u_i(x_{k'}, \mathbf{y}(x_{k'}) \geq u_i(x_k, \mathbf{y}(x_k))$ for all $i \in \mathbf{I}$ and all $(u_1, u_2) \in \mathcal{D}$, and the inequality is strict for at least one negotiator/utility function. Because $f^{\succeq_{\mathbf{y}}}$ is a logrolling mechanism, it must be that $x_k = \underset{[x_j, x_\ell]}{\boldsymbol{max}} \succeq_{\mathbf{y}}$, and so $x_k \succeq_{\mathbf{y}} x_{k'}$. Therefore, because $\mathcal{D}$ satisfies quid pro quo and $\succeq_{\mathbf{y}} \in \Pi_{\mathcal{D}}$, there must exist a negotiator $i \in \mathbf{I}$ satisfying $u_i(x_k, \mathbf{y}(x_k)) > u_i(x_{k'}, \mathbf{y}(x_{k'}))$ for all $u_i \in \mathcal{D}_i$ by condition $i.1$ of Definition 1 (strict inequality follows from the fact that u_i is strict), which yields the desired contradiction.

For step 2, we prove that any bundle in $\mathbf{B}\backslash\mathbf{B}^{\mathbf{y}}$ would make one of the negotiators worse off. To show this we suppose, for a contradiction, that there exists a bundle $(x_{k''}, y_r) \in \mathbf{B}\backslash\mathbf{B}^{\mathbf{y}}$, which is acceptable by both negotiators at type profile t, such that $u_i(x_{k''}, y_r) \geq u_i(x_k, \mathbf{y}(x_k))$ for all $i \in \mathbf{I}$ and all $(u_1, u_2) \in \mathcal{D}$, and the inequality is strict for at least one negotiator/utility function. Let $\mathbf{y}(x_k) = y_s$ and $\mathbf{y}(x_{k''}) = y_{s''}$. If $x_{k''} \notin [x_j, x_\ell]$, then it must be non-negotiable dealbreaker alternative for at least one of the negotiators at type profile t, and so by deal-breaker property $u^R_i(x_{k''}, y_r, t_i) < u^R_i(x_k, \mathbf{y}(x_k), t_i))$ holds for some $i \in \mathbf{I}$. Therefore, it must be that $x_{k''} \in [x_j, x_\ell]$. Because $f^{\succeq_{\mathbf{y}}}$ is a logrolling mechanism, $x_k = \underset{[x_j, x_\ell]}{\boldsymbol{max}} \succeq_{\mathbf{y}}$, and so $x_k \succeq_{\mathbf{y}} x_{k''}$. Because $\mathcal{D}$ satisfies quid pro quo and $\succeq_{\mathbf{y}} \in \Pi_{\mathcal{D}}$, there must exists a negotiator i satisfying $u_i(x_k, y_s) \geq u_i(x_{k''}, y_{s''})$ for all $u_i \in \mathcal{D}_i$ by condition $i.1$ of Definition 1. Suppose, without loss of generality, that the last inequality is true for Negotiator 1; namely $u_1(x_k, y_s) \geq u_1(x_{k''}, y_{s''})$ for all $u_1 \in \mathcal{D}_1$. Condition $i.1$ of Definition 1 also implies that $u_1(x_{k''}, y) \geq u_1(x_k, y)$ for all $u_1 \in \mathcal{D}_1$ and all $y \in \mathbf{Y}$, or equivalently $k'' \leq k$. Because each u_1 is decreasing, the last two inequalities imply $s \leq s''$. There are three exhaustive cases regarding the value of r in comparison to s and s'', and we consider each case next:

First, consider the case where $s \leq s'' \leq r$. Because $y_r \neq \mathbf{y}(x_{k''})$, it must be that $s'' < r$. Therefore, $u_1(x_{k''}, y_{s''}) > u_1(x_{k''}, y_r)$ for all $u_1 \in \mathcal{D}_1$ since u_1's are decreasing. The last inequality and $u_1(x_k, y_s) \geq u_1(x_{k''}, y_{s''})$ for all $u_1 \in \mathcal{D}_1$ yield $u_1(x_k, y_s) > u_1(x_{k''}, y_r)$ for all $u_1 \in \mathcal{D}_1$, contradicting that that both negotiators find bundle $(x_{k''}, y_r)$ better than (x_k, y_s). Second, consider the case where $r \leq s \leq s''$. Because the negotiators' preferences over alternatives are diametrically opposed and any $u_2 \in \mathcal{D}_2$ is increasing, it must be that for any $u_2 \in \mathcal{D}_2$, $u_2(x_k, y_s) \geq u_2(x_{k''}, y_s) \geq u_2(x_{k''}, y_r)$, one of which is strict because $(x_{k''}, y_r) \neq (x_k, y_s)$, contradicting that both negotiators find bundle $(x_{k''}, y_r)$ better than (x_k, y_s). Third, consider the case where $s < r < s''$. By condition $i.2$ of Definition 1, there is no $y_r \in \mathbf{Y}$ where $u_i(x_{k''}, y_r) \geq u_i(x_k, y_s)$ for all $i \in \mathbf{I}$ and $(u_1, u_2) \in \mathcal{D}$, contradicting again that both negotiators find bundle $(x_{k''}, y_r)$ better than (x_k, y_s). □

Lemma 8. *If the non-empty set $\mathcal{D} \subseteq \mathcal{U}$ satisfies quid pro quo and $f^{\succeq_{\mathbf{y}}}$ is a logrolling mechanism, where the partial order $\succeq_{\mathbf{y}}$ is in $\Pi_{\mathcal{D}}$, then $f^{\succeq_{\mathbf{y}}}$ is strategy-proof over $\mathcal{D}$.*

Proof. Assume that $\mathcal{D}$ satisfies quid pro quo and $f^{\succeq_{\mathbf{y}}}$ is a logrolling mechanism, where $\succeq_{\mathbf{y}} \in \Pi_{\mathcal{D}}$. Take any type profile $t = (x^1_\ell, x^2_j) \in \mathbf{T}$. We want to prove that no player has incentive to deviate from t. For this purpose, consider, without loss of generality, deviations of type x^1_ℓ of Negotiator 1, so fix j or x^2_j. If $\ell < j$, then $f^{\succeq_{\mathbf{y}}}_{\ell,j} = \phi$ and Negotiator 1 can change the outcome only if she deviates to a more accepting type (i.e., $x^1_{\ell'}$ where $\ell' > j > \ell$). In this case, $x^*_{[x_j, x_{\ell'}]} \in [x_j, x_{\ell'}]$, where $[x_j, x_{\ell'}] \cap \mathbf{N}(x^1_\ell) = \varnothing$, so $f^{\succeq_{\mathbf{y}}}_{\ell',j}$ is an unacceptable bundle for type x^1_ℓ. Therefore, deal-breaker property of the rationalizable utility functions implies that type x^1_ℓ of Negotiator 1 has no profitable deviation whenever $\ell < j$.

If $\ell = j$, then type x^1_ℓ of Negotiator 1 can deviate to a less accepting type or a more accepting type to change the outcome. In the first case, she deviates to a type $x^1_{\ell'}$ where $\ell' < \ell = j$, implying $f^{\succeq_{\mathbf{y}}}(x^1_{\ell'}, x^2_j) = f^{\succeq_{\mathbf{y}}}_{\ell',j} = \phi$. In the second case (i.e., she deviates to a type $x^1_{\ell''}$ where $\ell = j < \ell''$) it must be that $x^*_{[x_j, x_{\ell''}]} \in [x_j, x_{\ell''}]$, where $[x_j, x_{\ell''}] \cap \mathbf{N}(x^1_\ell) = \{x_\ell\}$, and so $f^{\succeq_{\mathbf{y}}}(x^1_{\ell''}, x^2_j) = f^{\succeq_{\mathbf{y}}}_{\ell'',j}$ is an unacceptable bundle for her, unless $f^{\succeq_{\mathbf{y}}}_{\ell'',j}$ is the same as $f^{\succeq_{\mathbf{y}}}_{\ell,j} = (x_\ell, \mathbf{y}(x_\ell))$. In any case, deal-breaker property of the rationalizable utility functions implies that type x^1_ℓ of Negotiator 1 has no profitable deviation whenever $\ell = j$.

Suppose now that $j < \ell$. By deal-breaker property, deviating to a type $x^1_{\ell'}$ where $\ell' < j < \ell$ is not profitable for type x^1_ℓ of Negotiator 1 because $f^{\succeq_{\mathbf{y}}}_{\ell',j} = \phi$. If she deviates to a type $x^1_{\ell''}$ where $j \leq \ell'' < \ell$, and if $f^{\succeq_{\mathbf{y}}}_{\ell,j} = (x_k, \mathbf{y}(x_k))$ and $f^{\succeq_{\mathbf{y}}}_{\ell'',j} = (x_{k''}, \mathbf{y}(x_{k''}))$, then by condition (iii) of Lemma 6 we must have $k'' \leq k$. Because the negotiators' utilities are monotonic and $\mathcal{D}$ satisfies logrolling, condition $i.1$ of Definition 1 implies $u_1(x_k, \mathbf{y}(x_k)) \geq u_1(x_{k''}, \mathbf{y}(x_{k''}))$ or equivalently $u_1(f^{\succeq_{\mathbf{y}}}_{\ell,j}) \geq u_1(f^{\succeq_{\mathbf{y}}}_{\ell'',j})$ for all $u_1 \in \mathcal{U}_1$. Thus, deviation to $x^1_{\ell''}$ can never be profitable for type x^1_ℓ of Negotiator 1 since all her rationalizable utility functions are identical to $\mathcal{U}_1 \in \mathcal{D}$ over all acceptable bundles. Finally, if she deviates to a more accepting type $x^1_{\ell'''}$ where $j \leq \ell < \ell'''$ and gets something different than $f^{\succeq_{\mathbf{y}}}_{\ell,j}$, then it must be that $x^*_{[x_j, x_{\ell'''}]} \in [x_{\ell+1}, x_{\ell'''}]$: To prove the last claim, suppose for a contradiction that $x^*_{[x_j, x_{\ell'''}]} \notin [x_{\ell+1}, x_{\ell'''}]$. Because $f^{\succeq_{\mathbf{y}}}$ is a logrolling mechanism, it must be that $x^*_{[x_j, x_{\ell'''}]} \in [x_j, x_{\ell'''}]$. The last two conditions imply $x^*_{[x_j, x_{\ell'''}]} \in [x_j, x_\ell]$. Because $f^{\succeq_{\mathbf{y}}}_{\ell,j} = \left(x^*_{[x_j, x_\ell]}, \mathbf{y}(x^*_{[x_j, x_\ell]})\right)$, it must be that $x^*_{[x_j, x_\ell]} \succeq_{\mathbf{y}} x^*_{[x_j, x_{\ell'''}]}$. Similarly, because $f^{\succeq_{\mathbf{y}}}_{\ell''',j} = \left(x^*_{[x_j, x_{\ell'''}]}, \mathbf{y}(x^*_{[x_j, x_{\ell'''}]})\right)$ and $[x_j, x_\ell] \subset [x_j, x_{\ell'''}]$, it must be that $x^*_{[x_j, x_{\ell'''}]} \succeq_{\mathbf{y}} x^*_{[x_j, x_\ell]}$, contradicting that $x^*_{[x_j, x_{\ell'''}]} \neq x^*_{[x_j, x_\ell]}$ and $\succeq_{\mathbf{y}}$ is antisymmetric. Therefore, it must be that $x^*_{[x_j, x_{\ell'''}]} \in [x_{\ell+1}, x_{\ell'''}]$, where $[x_{\ell+1}, x_{\ell'''}] \cap \mathbf{N}(x^1_\ell) = \varnothing$, and so $f^{\succeq_{\mathbf{y}}}_{\ell''',j}$ is an unacceptable bundle for type x^1_ℓ of Negotiator 1. Therefore, deviating to a more accepting type $x^1_{\ell'''}$ cannot not be profitable for her by the deal-breaker property of the rationalizable utility functions. Hence, type x^1_ℓ of Negotiator 1 has no profitable deviation whenever $j < \ell$.

Symmetric arguments would prove the same conclusion for Negotiator 2. Since we exhausted all possible deviations of type x^1_ℓ of Negotiator 1, we can conclude that $f^{\succeq_{\mathbf{y}}}$ is strategy-proof over $\mathcal{D}$. □

Proof of Theorem 2:

Proof of 'if': Assume that $\mathcal{D}$ satisfies quid pro quo and $f^{\succeq_\mathbf{y}}$ is a logrolling mechanism, where $\succeq_\mathbf{y} \in \Pi_\mathcal{D}$. The set $[x_j, x_\ell]$ is a connected subset of $\mathbf{X}$ whenever $j \leq \ell$, and $\succeq_\mathbf{y}$ is a semilattice for all connected subsets of $\mathbf{X}$. Therefore, $\underset{[x_j,x_\ell]}{\boldsymbol{max}} \succeq_\mathbf{y}$ uniquely exists whenever $j \leq \ell$. $f^{\succeq_\mathbf{y}}$ never suggests a non-negotiable dealbreaker alternative, and so, it is individually rational over $\mathcal{D}$. $f^{\succeq_\mathbf{y}}$ is efficient over $\mathcal{D}$ by Lemma 7 and strategy-proof over $\mathcal{D}$ by Lemma 8.

***Proof of 'only if'*:** Now assume that the mediation mechanism f is strategy-proof, efficient, and individually rational over $\mathcal{D}$. Theorem 1 implies an injective and decreasing function $\mathbf{y}$: $\mathbf{X} \to \mathbf{Y}$, a partial order $\succeq_\mathbf{y}$ on $\mathbf{X}$ such that $f = f^{\succeq_\mathbf{y}}$ is a logrolling mechanism. We need to prove that $\succeq_\mathbf{y} \in \Pi_\mathcal{D}$, and thus $\mathcal{D}$ satisfies quid pro quo.

To show that part $(1.i)$ of Definition 1 holds, let $x_k, x_{k'} \in \mathbf{X}$ be two distinct alternatives and $x_k \succeq_\mathbf{y} x_{k'}$. Suppose, without loss of generality, $k' < k$. Because any $u_1 \in \mathcal{D}_1$ is decreasing and $k' < k$, it must be that $u_1(x_{k'}, y) > u_1(x_k, y)$ for any $y \in \mathbf{Y}$. Strategy-proofness of f requires $U_1(f_{k,k'}, x_k^1) \geq U_1(f_{k',k'}, x_k^1)$ for all $U_1 \in \mathcal{D}_1^R$, and thus $u_1(f_{k,k'}) \geq u_1(f_{k',k'})$ for all $u_1 \in \mathcal{D}_1$ since each rationalizable and discernible utility function $U_1 \in \mathcal{D}_1^R$ of Negotiator 1 is identical to a discernible utility function $u_1 \in \mathcal{D}_1$ over the set of acceptable bundles. Recall the construction of $\succeq_\mathbf{y}$ in the proof of Theorem 1: $x_k \succeq_\mathbf{y} x_{k'}$ if and only if $f_{k,k'} = (x_k, \mathbf{y}(x_k))$. Therefore, the last inequality implies $u_1(x_k, \mathbf{y}(x_k)) > u_1(x_{k'}, \mathbf{y}(x_{k'}))$ because f is a logrolling mechanism (i.e., $f_{k',k'} = (x_{k'}, \mathbf{y}(x_{k'}))$ and x_k and $x_{k'}$ are two distinct alternatives. Thus, as required by part $(1.i)$ of Definition 1, for all $u_1 \in \mathcal{D}_1$ and all $y \in \mathbf{Y}$ we have $u_1(x_k, \mathbf{y}(x_k)) > u_1(x_{k'}, \mathbf{y}(x_{k'}))$ and $u_1(x_{k'}, y) > u_1(x_k, y)$.

To show that part $(1.ii)$ of Definition 1 holds, suppose for a contradiction that there is some $y \in \mathbf{Y}$ with $u_1(x_{k'}, y) > u_1(x_k, \mathbf{y}(x_k))$ and $u_2(x_{k'}, y) \geq U_2(x_k, \mathbf{y}(x_k))$ for all $(u_i, u_{-i}) \in \mathcal{D}$. Then, consider the type profile $t = (x_k^1, x_{k'}^2)$ where both x_k and $x_{k'}$ are negotiable by both types and $f(t) = (x_k, \mathbf{y}(x_k))$ (as discussed in the previous paragraph). The previous two inequalities imply that both these types prefer the bundle $(x_{k'}, y)$ over $f(t)$ for all $(u_1, u_2) \in \mathcal{D}$, contradicting that f is efficient over $\mathcal{D}$.

To show that part (2) of Definition 1 holds, recall that all sets of the form $[x_j, x_\ell]$ where $j, \ell \in \mathcal{I}$ and $j \leq \ell$ designate all the connected subsets of $\mathbf{X}$. Because $x^*_{[x_j,x_\ell]} = \underset{[x_j,x_\ell]}{\boldsymbol{max}} \succeq_\mathbf{y}$, it must be that every doubleton $\{x, x'\} \subseteq [x_j, x_\ell]$ has a least upper bound in $[x_j, x_\ell]$, denoted by $x^*_{[x_j,x_\ell]}$, and thus the poset $(S, \succeq_\mathbf{y})$ is a semilattice for all connected subset S of $\mathbf{X}$. Hence, $\succeq_\mathbf{y} \in \Pi_\mathcal{D}$ and $\mathcal{D}$ satisfies quid pro quo. ■

Proof of Theorem 3: Assume that $\mathbf{y} : \mathbf{X} \to \mathbf{Y}$ be an injective and decreasing function and $\succeq_\mathbf{y}$ be a partial order on $\mathbf{X}$. We first show $(i) \Rightarrow (ii)$. For this purpose, assume that $f_{\ell,j} = \left(x^*_{[x_j,x_\ell]}, \mathbf{y}(x^*_{[x_j,x_\ell]})\right)$ for all $\ell, j \in \mathcal{I}$ with $j \leq \ell$. Let $r_1 \in \mathcal{I}$ is such that $x_{r_1} = \underset{\mathbf{X}}{\boldsymbol{max}} \succeq_\mathbf{y}$. Therefore, it must be that $f_{r_1,r_1} = (x_{r_1}, \mathbf{y}(x_{r_1})) \in \mathbf{B}^\mathbf{y}$. Consider the matrix of f: All the entries on row r_1 to the left of entry f_{r_1,r_1}, all the entries on column r_1 below entry f_{r_1,r_1}, and all the entries in between must fill up with bundle $(x_{r_1}, t(x_{r_1}))$ because x_{r_1} has the highest rank over $\mathbf{X}$ and f always acts like the logrolling mechanism $f^{\succeq_\mathbf{y}}$ at these entries. Thus, the rectangle $\square^{r_1}_{m,1}$ fills up with $(x_{r_1}, \mathbf{y}(x_{r_1}))$. We let $\square^{r_1}_{m,1}$ be the first element of the rectangular partition of

$\triangle_{m,1}$. Note that, when $m \geq 3$, the so-far-unfilled $\triangle_{m,1} \backslash \square_{m,1}^{r_1}$ consists of at least one triangle (if $r_1 \in \{1, m\}$) and at most two triangles (if $r_1 \notin \{1, m\}$).

Next, take an arbitrary triangle $\triangle_{s,r} \in \triangle_{m,1} \backslash \square_{m,1}^{r_1}$. Note that either $s = r_1$ and $r = 1$, or $s = m$ and $r = r_1 + 1$. Let $f_{r_2,r_2} = (x_{r_2}, \mathbf{y}(x_{r_2})) \in \mathbf{B}^{\mathbf{y}}$ with $r_2 \neq r_1$ denote the logrolling bundle on the hypotenuse of $\triangle_{s,r}$ that satisfies $x_{r_2} = \underset{[x_s,x_r]}{\boldsymbol{max}} \succeq_{\mathbf{y}}$. Once again, starting from the hypotenuse of $\triangle_{s,r}$ all the so-far-unfilled entries on row r_2 to the left of entry f_{r_2,r_2}, all the so-far-unfilled entries on column r_2 below entry f_{r_2,r_2}, and all entries in between must fill up with bundle $(x_{r_2}, \mathbf{y}(x_{r_2}))$ because x_{r_2} has the highest rank among the alternatives in $[x_s, x_r]$. Thus, let $\square_{s,r}^{r_2}$ denote the second element of the rectangular partition of $\triangle_{m,1}$. Note that the so-far-unfilled set $\triangle_{m,1} \backslash \left\{\square_{m,1}^{r_1} \cup \square_{s,r}^{r_2}\right\}$ consists of at least one triangle. Iterate this reasoning and at each step pick a triangle from the so-far-unfilled subset of $\triangle_{m,1}$ and fill its corresponding rectangle with the bundle whose first component has the highest precedence with respect to $\succeq_{\mathbf{y}}$. By the finiteness of the problem, the rectangular partition is obtained in m steps.

Now we show $(ii) \Rightarrow (i)$. For this reason, we assume that the triangle $\triangle_{m,1}$ has a rectangular partition (denote it by $\mathcal{P}^1$) such that f assigns a unique bundle from the set of logrolling bundles $\mathbf{B}^{\mathbf{y}}$ to each rectangle in this partition. In this rectangular partition $\mathcal{P}^1$ of $\triangle_{m,1}(\equiv \triangle^1)$, let $\square^{r_1} \subset \triangle^1$ be the rectangle that includes the entry at the bottom left corner of triangle $\triangle^1$ (i.e., $f_{m,1}$). We construct the precedence order $\succeq_{\mathbf{y}}$ as follows: Let $f_{m,1}^{\mathbf{X}} = x_{r_1}$ have the higher precedence rank than any other alternative in $\mathbf{X}$; namely, $x_{r_1} \succeq_{\mathbf{y}} x$ for all $x \in \mathbf{X}$. Next consider $\triangle^1 \backslash \square^{r_1}$ which has a triangular partition $\mathcal{P}^2$ that consists of at most two triangles. Take an arbitrary triangle $\triangle^2 \in \mathcal{P}^2$ and let $\square^{r_2} \subset \triangle^2$ denote the rectangle that includes the entry at the bottom left corner of triangle $\triangle^2$, say $(x_{r_2}, \mathbf{y}(x_{r_2}))$. We let x_{r_2} have a higher precedence rank than any other alternative in $\mathbf{X}$ that appears on the hypotenuse of $\triangle^2$. Namely, if $r_2 < r_1$, then $x_{r_2} \succeq_{\mathbf{y}} f_{k,k}^{\mathbf{X}}$ for all $k \in \{1, ..., r_2 - 1, r_2 + 1, ..., r_1 - 1\}$, and if $r_2 > r_1$, then $x_{r_2} \succeq_{\mathbf{y}} f_{k,k}^{\mathbf{X}}$ for all $k \in \{r_1 + 1, ..., r_2 - 1, r_2 + 1, ..., m\}$. Iterate in this fashion by considering an arbitrary triangle from the remaining partition $\triangle^1 \backslash \{\square^{r_1}, \square^{r_2}\}$. At the end of this finite procedure (consisting of exactly m steps), we obtain a transitive, antisymmetric but possibly incomplete strict precedence order $\succeq_{\mathbf{y}}$ on $\mathbf{B}^{\mathbf{y}}$. Moreover, by construction we have $f_{\ell,j} = \left(x^*_{[x_j,x_\ell]}, \mathbf{y}(x^*_{[x_j,x_\ell]})\right)$ where $x^*_{[x_j,x_\ell]} = \underset{[x_j,x_\ell]}{\boldsymbol{max}} \succeq_{\mathbf{y}}$ for all $\ell, j \in \mathcal{I}$ with $j \leq \ell$. This completes the proof. ■

Proof of Theorem 4: Constrained Shortlisting mechanism clearly belongs to a logrolling mechanisms family. Fix the set of logrolling bundles $\mathbf{B}^{\mathbf{y}} = \{(x, \mathbf{y}(x)) | x \in \mathbf{X}\}$ and the family of logrolling mechanisms whose range is $\mathbf{B}^{\mathbf{y}} \cup \{\phi\}$. Let $b_j = (x_j, \mathbf{y}(x_j)) \in \mathbf{B}^{\mathbf{y}}$ denote a logrolling bundle. To see that the rank variance of a CS mechanism is lower than any other member of the logrolling mechanism family, we simply consider two cases about the number of possible alternatives.

First, when m is odd, $var(b_k) = (m+1)^2$. For any $b_{k-j}, b_{k+j} \in \mathbf{B}^{\mathbf{y}}$ with $j < k$, we have $var(b_{k-j}) = var(b_{k+j}) = 2(\frac{(m+1)}{2} - j)^2 + 2(\frac{(m+1)}{2} + j)^2 = (m+1)^2 + 4j^2$. Thus, $var(b_k) < var(b)$ for any $b \in \mathbf{B}^{\mathbf{y}} \backslash \{b_k\}$. Since any member of the logrolling mechanism family must pick an element of $\mathbf{B}^{\mathbf{y}}$ whenever the mutual zone of agreement is non-empty (by Theorem 1), minimization of rank variance requires that $x_k \succeq_{\mathbf{y}}^{CS} x$ for any $x \in \mathbf{X}$. Also observe that $var(b_k) < var(b_{k-1}) <$

$\ldots < var(b_1)$ and $var(b_k) < var(b_{k+1}) < \ldots < var(b_m)$. Thus, minimization of rank variance subsequently requires that $x_{k-1} \succeq^{CS}_{\mathbf{y}} \ldots \succeq^{CS}_{\mathbf{y}} x_1$ and $x_{k+1} \succeq^{CS}_{\mathbf{y}} \ldots \succeq^{CS}_{\mathbf{y}} x_m$. Note that when m is odd, rank variance of the unique CS mechanism is strictly less than any other member of the logrolling mechanisms family.

Second, when m is even, $var(b_{\overline{k}}) = var(b_{\underline{k}}) = \frac{1}{2}(m^2 + (m+2)^2)$. For any $b_{\underline{k}-j}, b_{\overline{k}+j} \in \mathbf{B}^{\mathbf{y}}$ with $j < k$, we have $var(b_{\underline{k}-j}) = var(b_{\overline{k}+j}) = 2(\frac{m}{2} - j)^2 + 2(\frac{(m+2)}{2} + j)^2 = \frac{1}{2}(m^2 + (m+2)^2) + 4j^2$. Hence, $var(b_{\overline{k}}) = var(b_{\underline{k}}) < var(b)$ for any $b \in \mathbf{B}^{\mathbf{y}} \backslash \{b_{\overline{k}}, b_{\underline{k}}\}$. Note that we also have $var(b_{\underline{k}}) = var(b_{\overline{k}}) < var(b_{\underline{k}-1}) < \ldots < var(b_1)$ and $var(b_{\underline{k}}) = var(b_{\overline{k}}) < (b_{\overline{k}+1}) < \ldots < var(b_m)$. Then, minimization of rank variance subsequently requires that either $x_{\overline{k}} \succeq^{CS}_{\mathbf{y}} x_{\underline{k}}$ or $x_{\underline{k}} \succeq^{CS}_{\mathbf{y}} x_{\overline{k}}$ together with $x_{\underline{k}-1} \succeq^{CS}_{\mathbf{y}} \ldots \succeq^{CS}_{\mathbf{y}} x_1$ and $x_{\overline{k}+1} \succeq^{CS}_{\mathbf{y}} \ldots \succeq^{CS}_{\mathbf{y}} x_m$. Note that when m is even, rank variance of a CS mechanism is weakly less than any other member of the logrolling mechanisms family. ■

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