Are Myersonian Common-Knowledge Events Common Knowledge?

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Abstract

This note presents a microfoundation for the belief-based common knowledge concept proposed by Holmström and Myerson (1983) and further refined by Myerson (1997), by showing its equivalence to the hierarchical description of common knowledge, agent i knows that j knows that ... knows event E. Furthermore, it provides a useful algorithm to find common knowledge events.

1 Introduction

The idea of common knowledge is a central concept in game theory. An event E is common knowledge among a set of agents if each agent knows E, each agent knows every other agent knows E, and so on *ad infinitum*, as articulated informally by Schelling (1960), and more precisely by Lewis (1969) and Schiffer (1972). We refer to this characterization as the *hierarchical description* of common knowledge. In the economics literature, Aumann (1976) was the first to formally define common knowledge in a framework where private information is represented by partitions of some state space, and to relate it to the hierarchical description. In a

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seminal paper, Holmström and Myerson (1983) introduced a tractable description for common knowledge, based on agents' prior beliefs, which has been used in many papers in a variety of contexts: see for example Crawford (1985, p. 821) and Lee (2019) in the context of efficient and durable decision rules, and Vohra (1999), Serrano, Vohra and Volij (2001, p. 1692) and Forges, Minelli and Vohra (2002, p. 25) in the context of the core. Myerson (1997) refined the Holmström-Myerson notion, and in this note we provide a microfoundation for *Myersonian common knowledge* by showing that it is equivalent to the hierarchical description of common knowledge.

For this purpose, we first provide a belief-based description of knowledge in an environment we borrow from Holmström and Myerson (1983) where the state space T is rectangular, and each agent observes only her own type. We say that an agent i *knows* an event $E \subseteq T$ in some state t in T if given her type t_i, she is sure that the state t is in E. We then say that an event E is *self-evident* (i.e., mutual knowledge) if all agents know it whenever it occurs. Self-evident events are crucial for our equivalence results. Theorem 1 establishes the equivalence of self-evident and Myersonian common knowledge events. Theorem 2 shows that an event is Myersonian common knowledge if and only if that event is common knowledge in every state in it. Thus, Theorem 2 provides an affirmative answer to the question posed in the title.

However, the fact that an event is common knowledge is not easy to verify due to the infinite hierarchy of knowledge. To provide such a verification, we iteratively define the *event generated by a state* t, denoted by $S^{\infty}(t)$, by first collecting the set of states any agent believes possible in state t, denoted by $S^0(t)$, and then collecting all states any agent believes possible in some state in $S^0(t)$, denoted by $S^1(t)$, and then collecting all states any agent believes possible in some state in $S^1(t)$, denoted by $S^2(t)$, and so on ad infinitum. Having constructed $S^{\infty}(t)$ as above, Theorem 3 fully characterizes and computes common knowledge events. In particular, we show that checking E is common knowledge in state t is equivalent to checking if $S^{\infty}(t)$ is contained in E.

The note is structured as follows: Section 2 presents the results, Section 3 presents the proofs, and finally the Appendix relates Myersonian common knowledge to the concepts conventional in the literature through the formulations of

Brandenburger and Dekel (1987) that, in admitting zero probability events, generalize the notion used in Aumann (1976).

2 Results

We begin by describing the primitives of the information structure due to Holmström and Myerson (1983). For expositional simplicity we present a two-agent setting with common prior beliefs, but all our results generalize immediately to the case of more than two individuals with non-common priors.

Definition 1. *An* **information structure** *is defined by* $I = \{N, T_1, T_2, p\}$*, where*

(i) $N = \{1, 2\}$ *denotes the set of agents,*

(ii) T_i *denotes the finite set of (private) types of agent* $i \in N$*,*

(iii) p *denotes agents' common prior probability distribution on* $T = T_1 \times T_2$*.*

Each agent i is privately informed about her type $t_i \in T_i$, but does not know the type of the other agent $t_{-i} \in T_{-i}$, and hence does not know the true **state** $t \in T$.

Each agent has two sources of information that form the basis of her knowledge about the true state (here we use the terms information and knowledge loosely): (1) Agent $i \in N$ with type $t_i \in T_i$ knows that the true state is some $r \in T$ such that $t_i = r_i$, and (2) she is sure that the true state is not some $t \in T$ with $p(t) = 0$. Naturally, knowledge of an agent should be a combination of these two sources of information, and we formally define it below after some additional preliminary concepts.

Given the prior p, with slight abuse of the notation, we denote the marginal probability of type $t_i \in T_i$ by

$$p(t_i) \equiv \sum_{t_{-i} \in T_{-i}} p(t_i, t_{-i}). \tag{1}$$

Consider a possible scenario where agent i's true type is t_i, and the prior p is such that $p(t_i) = 0$. Namely, agent i knows that her type is t_i, and she is sure that her type is not t_i. To eliminate such anomalies, we assume that the information structure is *consistent*. Throughout this paper we impose the following assumption.

Assumption 1. *The information structure $I = \{N, T_1, T_2, p\}$ is* ***consistent*** *in the following sense:* $p(t_i) > 0 \quad$ *for all* $\ t_i \in T_i$ *and* $i \in N$.

The consistency assumption does not imply full support for the common prior, but guarantees that the conditional probabilities, defined by

$$p(t_{-i}|\ t_i) = \frac{p(t_i, t_{-i})}{p(t_i)},$$

are always well-defined.

An **event** $E \subseteq T$ is a non-empty set of states. Agent i knows an event E in some state t means that i is sure, in state t, that the true state is an element of E. More formally:

Definition 2. *Agent i* ***knows event*** *E* ***in state*** *$t = (t_i, t_{-i}) \in T$ if for every $r_{-i} \in T_{-i}$,*

$$(t_i, r_{-i}) \notin E \ \ \textit{implies} \ \ p(r_{-i}|t_i) = 0.$$

The **knowledge function** K_i of agent i maps any event $E \subseteq T$ to a subset $K_i(E)$ of T consisting of all states in which agent i knows the event E.

Definition 3. *For any event $E \subseteq T$,*

$$K_i(E) = \Big\{ t \in T \ \Big| (t_i, r_{-i}) \notin E \ \textit{implies} \ p(r_{-i}|t_i) = 0 \Big\}.$$

We adopt a belief-based definition of knowledge function, as opposed to the partition-based definition as in Aumann (1976), since our goal is to link it to the belief-based notion of common knowledge due to Myerson (1997) formalized below. This knowledge function uses the information structure that we begin with and incorporates both sources of information agents have. As we show in the Appendix, it is equivalent to the belief-based knowledge function of Brandenburger and Dekel (1987) restricted to our setting.

The following is the standard hierarchical description of common knowledge.

Definition 4. *An event $E \subseteq T$ is* ***common knowledge in state*** *$t \in T$ if t is a member of every set in the infinite sequence $K_1(E)$, $K_2(E)$, $K_1(K_2(E))$, $K_2(K_1(E))$, ...*

Remark 1. If $E \subseteq F \subseteq T$, then $K_i(E) \subseteq K_i(F)$ for all $i \in N$. As such, if $E \subseteq F$ and E is common knowledge in state t, then $t \in K_i(F) \cap K_i(K_j(F)) \cap$

$K_i(K_j(K_i(F))) \cap \ldots$, that is, if an event is common knowledge in some state t, all its supersets are also common knowledge in that state.

In order to present Myerson's definition of common knowledge events, we need one more piece of notation. For any event $E \subseteq T$, let E_i denote the set of types of agent i that is consistent with the state being in E. Formally, $E_i = \{t_i | t \in E\}$.

Definition 5 (Myerson 1997). *An event $E \subseteq T$ is **Myersonian common-knowledge** if for all $i \in N$, all $t_i \in E_i$ and all $r_{-i} \in T_{-i}$,*

$$(t_i, r_{-i}) \notin E \ \ \textit{implies} \ \ p(r_{-i}|t_i) = 0.$$

The precursor of this definition dates back to Holmström and Myerson (1983) which differs from the definition above by requiring that event E is of the form $E = E_1 \times E_2$. However, this requirement can be unreasonably demanding as can be seen in the following remark.

Remark 2. Under Assumption 1, an event $E \subseteq T$ is common knowledge in the sense of Holmström and Myerson if and only if $E = T$.

The proof of this claim and all other proofs are deferred to Section 3.

Myerson's definition is unconventional in the sense that it is not conditional on the true state. Namely, if an event is Myersonian common knowledge, then it is so regardless of the realized state. However, it incorporates agents' private information and prior beliefs, and so if agent i's private type is t_i, then she is sure that the true state, of the form (t_i, r_{-i}), is in E. Thus, Myersonian common knowledge is not without any justification. Nevertheless, the question "is there a hierarchical microfoundation for the Myersonian common knowledge?" still begs for an answer. We address this question next, and for this purpose, introduce the notion of self-evident events, which is standard in the literature (see, for example, Osborne and Rubinstein (1994)).

Definition 6. *An event $E \subseteq T$ is **self-evident** if $E \subseteq K_i(E)$ for all $i \in N$.*

In the next result, we establish the equivalence between Myersonian common knowledge events and self-evident events.

Theorem 1. *An event $E \subseteq T$ is Myersonian common-knowledge if and only if E is self-evident.*

The intuition for Theorem 1 is simple: An event E is Myersonian common-knowledge if and only if each agent is sure, in any state $t \in E$, that the true state is in E, meaning that both agents know the event E in every state t in E, and thus, E is self-evident. The answer for our inquiry "Are Myersonian common-knowledge events common knowledge?" is affirmative in the sense we formally state below.

Theorem 2. *An event $E \subseteq T$ is Myersonian common-knowledge if and only if E is common knowledge in every state $t \in E$.*

The intuition of Theorem 2 is also straightforward once we establish the following instrumental observation; an event E is self-evident if and only if $E \subseteq K_i(F)$ for all $i \in N$ and all events $F \supseteq E$. By Theorem 1, an event E is Myersonian common-knowledge if and only if it is self-evident. Therefore, applying our observation repeatedly on E proves that E is a subset of $K_i(E)$, $K_j(K_i(E))$, $K_i(K_j(K_i(E)))$, $\ldots$, and hence, E is common knowledge in every state t in E.

An immediate implication of Theorems 1 and 2 is the following:

Corollary 1. *An event E is self evident if and only if E is common knowledge in every state $t \in E$.*

As mentioned earlier, Myersonian common knowledge is a state independent concept. For this reason, its direct comparison with the hierarchical description of common knowledge is not straightforward. A common knowledge event need not be Myersonian common knowledge. Conversely, a Myersonian common knowledge event E may not be common knowledge in state t that is outside of E. The following example illustrates these points.

Example 1. Consider the following information structure with $N = \{1, 2\}$, $T_1 = \{a, b\}$, $T_2 = \{A, B\}$ and the common priors are given by the following matrix where zeros indicate null states and plus signs indicate non-null states:

	A	B
a	$+$	0
b	0	$+$

As we will verify later, event $E = \{(a, A), (b, A)\}$ is common knowledge in state (a, A), but not Myersonian common knowledge; conversely, event $E' = \{(a, A)\}$ is Myersonian common knowledge, but not common knowledge in state (b, B).

Verifying that an event is common knowledge is not easy due to infinite hierarchy of knowledge. We now provide an algorithm to fully characterize and compute (Myersonian) common knowledge events. In light of Remark 1, in order to characterize the events that are common knowledge in some state t, it is sufficient to characterize the smallest event F that is common knowledge in that state. Therefore, we look for the smallest set F such that $t \in K_i(F) \cap K_i(K_j(F)) \cap K_i(K_j(K_i(F))) \cap \ldots$.

For this purpose, fix the state $t = (t_1, t_2)$ and consider all the events F' that each agent i knows in state t (i.e., $t \in K_i(F')$ for all $i \in N$). Call the smallest of these events $S^0(t)$. Thus, $S^0(t)$ is the smallest event that both agents know in state t, which, by Definition 3, must include all the positive probability states (t_i, r_{-i}) for each $i \in N$ and $r_{-i} \in T_{-i}$.

Next, consider all the events that both agents know and that both agents know that they know in state t: Namely the events F'' such that $t \in K_i(F'') \cap K_i(K_j(F''))$ for all $i, j \in N$ with $j \neq i$. Take the smallest of these events and call it $S^1(t)$. If agents know event $S^1(t)$ in state t, then they will know and also know that they know all supersets of this event in state t. Thus, an event that is not a superset of $S^1(t)$ cannot be common knowledge in state t. By construction $S^1(t) \supseteq S^0(t)$. As we iterate this process, we create an infinite sequence of increasingly-nested sets $\{S^k(t)\}$. Because all these sets are subsets of the finite state space T and they are increasingly nested, this sequence must converge to a fixed set after some k. We call this limiting set $S^\infty(t)$, which is analogous to the event "reachable from state t" in the context of partitional information structure of Aumann (1976). The next definition formalizes this construction.

Definition 7. *For any state $t \in T$, the* ***event generated by state*** *t, denoted by $S^\infty(t)$, is defined recursively as follows:*

1. *For each $i \in N$, define $T(t_i) = \{(t_i, r_{-i}) \in T | \ p(r_{-i}|t_i) > 0\}$ to be the set of all states that agent i believes possible, conditional on her type being t_i.*

2. *Define $T(t) = \bigcup_{i \in N} T(t_i)$ to be the set of all states that either agent believes possible, conditional on type state being t.*

3. *For any event $E \subseteq T$, define $T(E) = \bigcup_{t \in E} T(t)$ to be the set of all states that either agent believes possible, conditional on state being some $t \in E$.*

4. *Set* $S^0(t) = T(t)$*, and for any integer* $k \geq 0$*, recursively set*

$$S^{k+1}(t) = T(S^k(t)).$$

5. *Finally, define*

$$S^\infty(t) = \bigcup_{k\geq 0} S^k(t).$$

Remark 3. *For any* $t \in T$*,* $S^\infty(t)$ *is a self-evident event, and for any* $k \in \mathbb{N}$*,* $S^k(t)$ *is a non-empty subset of* $S^{k+1}(t)$*.*

Because T is a rectangular state space, $S^0(t)$ consists of all positive probability states that are in the same row and column with t. For any $k \geq 0$, $S^{k+1}(t)$ consists of all positive probability states that are in the same row and column with some state in $S^k(t)$. The following example illustrates the construction of $S^\infty(t)$.

Example 2. Consider the following information structure with two agents, each of which has three types:

	A	B	C
a	0	0	+
b	0	+	0
c	+	+	0

Suppose that $t = (a, A)$. In this case $S^0(t) = T(t) = \{(a, C), (c, A\}$. Finding this set is simple, it is the collection of all non-null states that appear on the first row or first column because t is in the first row and first column. Then, we have $S^1(t) = T(S^0(t)) = \{(a, C), (c, A), (c, B)\}$. This set is the collection of all non-null states that appear on the first column or third row (because (c, A) is on the intersection of row 3 and column 1) and all non-null states that appear on the first row or third column (because (a, C) is on the intersection of row 1 and column 3). Continuing in this manner we get $S^k(a, A) = \{(a, C), (c, A), (c, B), (b, B)\}$ for all $k = 2, 3, ...$, and hence $S^\infty(a, A) = \{(a, C), (c, A), (c, B), (b, B)\}$. Likewise, $S^\infty(b, C) = S^\infty(c, C) = \{(a, C), (c, A), (c, B), (b, B)\}$. However, we have $S^\infty(a, C) = \{(a, C)\}$ and $S^\infty(b, B) = \{(b, B), (c, B), (c, A)\}$.

Having constructed $S^\infty(t)$, we can check if an event E is common knowledge in state t by simply checking if $S^\infty(t)$ is contained in E. In fact, we can also use

$S^\infty(t)$ to check if event E is common knowledge in any other state in $S^\infty(t)$. The following theorem formalizes these observations.

Theorem 3. *For any event $E \subseteq T$ and state $t \in T$, the following three statements are equivalent:*

1. *Event E is common knowledge in state t.*
2. *Event E is common knowledge in any state $s \in S^\infty(t)$.*
3. *$S^\infty(t) \subseteq E$.*

The next result immediately follows from Theorems $1-3$. It is useful in finding self-evident events, or equivalently Myersonian common knowledge events.

Corollary 2. *An event E is self-evident (equivalently, Myersonian common knowledge) if and only if $S^\infty(t) \subseteq E$ for all $t \in E$.*

We now have the tools to verify the claims we made in Example 1.

Example 1 (Revisited). In this example, event $E = \{(a, A), (b, A)\}$ is common knowledge in state (a, A) since $S^\infty(a, A) = \{(a, A)\} \subseteq E$, but not Myersonian common knowledge since $S^\infty(b, A) = \{(a, A), (b, B)\} \nsubseteq E$; conversely, $E' = \{(a, A)\}$ is Myersonian common knowledge since $S^\infty(a, A)$ is a subset of E', but not common knowledge in state (b, B) since $S^\infty(b, B) = \{(b, B)\} \nsubseteq E$.

3 Proofs of the Results

We begin with the proof of the argument in Remark 2.

Proof of Remark 2. Note that an event E is common knowledge in the sense of Holmström and Myerson [HM henceforth] iff E is of the form $E = E_1 \times E_2$, where $E_i \subseteq T_i$ for all $i \in N$, and $p(r_{-i}|t_i) = 0$ for all $t \in E$, $r \notin E$, and $i \in N$. It is easy to verify that T is common knowledge in the sense of HM. Now pick an event $E \subseteq T$ that is common knowledge in the sense of HM. Fix some $t \in E$ and consider any $i \in N$. By Assumption 1, there must exist some $r^*_{-i} \in T_{-i}$ such that $p(t_i, r^*_{-i}) > 0$ (i.e., $p(r^*_{-i}|t_i) > 0$). Because $t \in E$ and E is common knowledge in the sense of HM, we must have $(t_i, r^*_{-i}) \in E$. For the similar reasons, (t'_i, r^*_{-i}) must be in E for every $t'_i \in T_i$, and thus, $E_i = T_i$. Because $i \in N$ was arbitrary and E is of the form $E = E_1 \times E_2$, we must have $E = T$. □

We now present some technical results that we use later.

Lemma 1. *For any event $E \subseteq T$,*

$$K_i(E) = \Big\{ t \in T \;\Big|\; \text{if } (t_i, r_{-i}) \notin E, \text{ then } p(t_i, r_{-i}) = 0 \Big\}.$$

Proof. Follows immediately from Assumption 1 and Definition 3. □

Lemma 2. *For any $t \in T$, $s \in S^\infty(t)$, $i \in N$, and $r_{-i} \in T_{-i}$, we have $(s_i, r_{-i}) \in S^\infty(t)$ if and only if $p(r_{-i}|s_i) > 0$.*

Proof. Fix $t \in T$, $s \in S^\infty(t)$, $i \in N$ and $r_{-i} \in T_{-i}$. Since $s \in S^\infty(t)$, there exists k such that $s \in S^k(t)$. Suppose first $p(r_{-i}|s_i) > 0$. Then $(s_i, r_{-i}) \in T(s_i) \subset T(s) \subset T(S^k(t))$, and hence $(s_i, r_{-i}) \in S^\infty(t)$.

Suppose now $(s_i, r_{-i}) \in S^\infty(t)$. If $(s_i, r_{-i}) \in T(s_i)$, then $p(r_{-i}|s_i) > 0$. Suppose $(s_i, r_{-i}) \notin T(s_i)$. Then we must have $p(r_{-i}|s_i) = 0$. Since $(s_i, r_{-i}) \in S^\infty(t)$, there exists k' such that $(s_i, r_{-i}) \in S^{k'}(t) = T(S^{k'-1}(t))$. Then, there exists $s' \in S^{k'-1}(t)$ such that $(s_i, r_{-i}) \in T(s')$. This in turn implies that there exists $j \in N$ such that $(s_i, r_{-i}) \in T(s'_j)$. Since $p(r_{-i}|s_i) = 0$, it must be the case that $j \neq i$, $r_j = s'_j$ and $p(s_i|r_j) > 0$. Since $p(s_i|r_j)p(r_j) = p(r_{-i}|s_i)p(s_i)$, we must have $p(r_j) = 0$. This violates Assumption 1. Thus, $(s_i, r_{-i}) \in T(s_i)$, and so $p(r_{-i}|s_i) > 0$ as desired. □

Lemma 3. *For any event $E \subseteq T$, agent $i \in N$, and states $t, s \in T$ with $s_i = t_i$ and $p(s_{-i}|t_i) > 0$,*

(i) if $t \in K_i(E)$, then $s \in K_i(E) \cap E$;

(ii) if E is common knowledge in state t, then E is common knowledge in state s.

Proof. Fix event $E \subseteq T$, agent $i \in N$, and states $t, s \in T$ with $s_i = t_i$ and $p(s_{-i}|t_i) > 0$. *(i):* Suppose $t \in K_i(E)$. By the definition of $K_i(E)$, we have $p(r_{-i}|t_i) = 0$ for all $(t_i, r_{-i}) \notin E$. This has two immediate implications. First, since $p(s_{-i}|t_i) > 0$, we must have $s \in E$. Second, since $s_i = t_i$, we have $p(r_{-i}|s_i) = 0$ for all $(s_i, r_{-i}) \notin E$, and therefore $s \in K_i(E)$ by the definition of $K_i(E)$. Thus $s \in K_i(E) \cap E$.

(ii) Suppose E is common knowledge in state t. Then t is a member of every set in the sequence $K_i(E), K_i(K_j(E)), K_i(K_j(K_i(K_j(E)))), \ldots$. Then by part (i),

$s \in K_i(E) \cap E$, $s \in K_i(K_j(E)) \cap K_j(E)$, $s \in K_i(K_j(K_i(K_j(E)))) \cap K_j(K_i(K_j(E)))$, and so on. Thus, E is common knowledge in state s. □

Lemma 4. *For all $i \in N$ and events $E, F \subseteq T$, if $E \subseteq F$, then $K_i(E) \subseteq K_i(F)$.*

Proof. Take any $i \in N$ and $t \in K_i(E)$. For any $r_{-i} \in T_{-i}$, if $(t_i, r_{-i}) \notin F$, then $(t_i, r_{-i}) \notin E$. Since $t \in K_i(E)$, $(t_i, r_{-i}) \notin E$ requires that $p(r_{-i}|t_i) = 0$. Hence, t must be in $K_i(F)$. □

Lemma 5. *If $E \subseteq F$ and E is a self-evident event, then $E \subseteq K_i(F)$ for all $i \in N$.*

Proof. Since $E \subseteq F$, Lemma 4 implies that $K_i(E) \subseteq K_i(F)$ for all $i \in N$. Since E is self-evident, $E \subseteq K_i(E)$ for all $i \in N$. □

Proof of Theorem 1. Suppose E is Myersonian common-knowledge. Take any $t \in E$. Because E is Myersonian common-knowledge, we have $p(r_{-i}|t_i) = 0$ for any $i \in N$, and $r_{-i} \in T_{-i}$ with $(t_i, r_{-i}) \notin E$. This implies $t \in K_i(E)$, and hence $E \subseteq K_i(E)$ for all $i \in N$.

Now, suppose E is self-evident, that is, $E \subseteq K_i(E)$ for all $i \in N$. Take any $t \in E$, $i \in N$, and $r_{-i} \in T_{-i}$ such that $(t_i, r_{-i}) \notin E$. Since $t \in E \subseteq K_i(E)$, we have $p(r_{-i}|t_i) = 0$. Hence, E is Myersonian common-knowledge. □

Proof of Theorem 2. Assume E is a Myersonian common knowledge event. By Theorem 1, E is self-evident, that is $E \subseteq K_i(E)$ for all $i \in N$. Then by Lemma 5, $E \subseteq K_j(K_i(E))$ for all $i, j \in N$. Applying Lemma 5 one more time, we obtain $E \subseteq K_i(K_j(K_i(E)))$ for all $i, j \in N$. Repeating this logic ensures that E is a subset of every set in the infinite sequence $K_i(E)$, $K_j(E)$, $K_i(K_j(E))$, $K_j(K_i(E))$, ... Thus, event E is common knowledge in any state $t \in E$.

Next, assume E is common knowledge in every state $t \in E$. By definition, $t \in K_i(E)$ for all $i \in N$ and all $t \in E$. Thus, $E \subseteq K_i(E)$ for all $i \in N$, which means E is self-evident. Then by Theorem 1, E is Myersonian common knowledge. □

Proof of Remark 3. It is rather straightforward to verify that for any $t \in T$ and $k \in \mathbb{N}$, $S^k(t)$ is a non-empty subset of $S^{k+1}(t)$. It remains to show that for all $t \in T$, $S^\infty(t)$ is self-evident. Towards this end pick $t \in T$. $S^\infty(t)$ is a non-empty subset of T for any $t \in T$ because $S^k(t)$ is non-empty for all $k \in \mathbb{N}$. Moreover, by

Lemma 2, for any $s \in S^\infty(t)$, $i \in N$, and $r_{-i} \in T_{-i}$, we have $(s_i, r_{-i}) \in S^\infty(t)$ if and only if $p(r_{-i}|s_i) > 0$. Therefore, for any $s \in S^\infty(t)$, $i \in N$, and $r_{-i} \in T_{-i}$, if $(s_i, r_{-i}) \notin S^\infty(t)$, then $p(r_{-i}|s_i) = 0$, implying that $s \in K_i(S^\infty(t))$. Thus, $S^\infty(t)$ is self-evident. □

Proof of Theorem 3. Fix event $E \subseteq T$ and state $t \in T$. We show (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (2): Suppose event E is common knowledge in state t. Take any $s \in S^\infty(t)$. Then there exists k such that $s \in S^k(t)$. We will show that E is common knowledge in state s. We will prove this claim inductively on k.

Base case: Suppose $k = 0$, i.e. $s \in S^0(t)$. Then, by the definition of $S^0(t)$, there exists an agent $i \in N$ such that $s_i = t_i$ and $p(s_{-i}|t_i) > 0$. Since E is common knowledge in state t, part (ii) of Lemma 3 implies that E is common knowledge in state s. Therefore, E is common knowledge in any state in $S^0(t)$.

Induction step: Suppose E is common knowledge in any state in $S^{k-1}(t)$. We will show that E is common knowledge in any state $S^k(t)$. By Remark 3, $S^{k-1}(t) \subseteq S^k(t)$. If $S^k(t) = S^{k-1}(t)$, then the result is immediate. Otherwise, pick any s^k in $S^k(t) \setminus S^{k-1}(t)$. By construction, there exists $s^{k-1} \in S^{k-1}(t)$ and $i \in N$ with $s_i^k = s_i^{k-1}$ and $p(s_{-i}^k|s_i^{k-1}) > 0$. Since E is common knowledge in any state in s^{k-1}, part (ii) of Lemma 3 implies that E is common knowledge in state s^k.

(2) ⇒ (3): Suppose event E is common knowledge in any state $s \in S^\infty(t)$. Take any $s \in S^\infty(t)$. Then, $p(s_{-i}|s_i) > 0$ by Lemma 2 and $s \in E$ by Lemma 3, Hence, we must have $S^\infty(t) \subseteq E$.

(3) ⇒ (1): Suppose $S^\infty(t) \subseteq E$. First we show that E is common knowledge in any state $s \in S^\infty(t)$. By Remark 3, $S^\infty(t)$ is self-evident. Since $S^\infty(t) \subseteq E$, Lemma 5 implies $S^\infty(t) \subseteq K_i(E)$ for all $i \in N$. By applying Lemma 5 one more time and using that $S^\infty(t)$ is self-evident, we obtain $S^\infty(t) \subseteq K_j(S^\infty(t)) \subseteq K_j(K_i(E))$ for all $i, j \in N$. Repeating this logic ensures that $S^\infty(t)$ is contained in every set in the infinite sequence $K_1(E)$, $K_2(E)$, $K_1(K_2(E))$, $K_2(K_1(E))$, ... Hence, event E is common knowledge in any state $s \in S^\infty(t)$.

Next we show that E is common knowledge in state t. Take any $i \in N$ and $r_{-i} \in T_{-i}$. If $(t_i, r_{-i}) \notin E$, then $(t_i, r_{-i}) \notin S^\infty(t)$, and thus $p(r_{-i}|t_i) = 0$ by Lemma 2, implying that $t \in K_i(E)$. For $j \in N$ with $j \neq i$, if $(t_i, r_{-i}) \notin K_j(E)$, then

$(t_i, r_{-i}) \notin S^\infty(t)$ because $S^\infty(t) \subseteq K_j(E)$, and thus $p(r_{-i}|t_i) = 0$, implying that $t \in K_i(K_j(E))$. Similarly, if $(t_i, r_{-i}) \notin K_j(K_i(E))$, then $(t_i, r_{-i}) \notin S^\infty(t)$ because $S^\infty(t) \subseteq K_j(K_i(E))$, and thus $p(r_{-i}|t_i) = 0$, implying that $t \in K_i(K_j(K_i(E)))$. Repeating these arguments ensures that t is also an element of every set in the infinite sequence $K_i(E)$, $K_j(E)$, $K_i(K_j(E))$, $K_j(K_i(E))$, ... Hence, E is common knowledge in state t. □

Appendix

A The Knowledge Function

Brandenburger and Dekel (1987) [BD henceforth] generalize the information based common knowledge definition of Aumann (1976) by allowing null events and provide an equivalent belief based definition for common knowledge in a Bayesian framework. Their framework is more general than the one we use in this note: they start with a probability space – a set of possibly uncountable states, a σ-field, and a common prior p – and use a regular conditional probability function to represent each agent's posterior belief about the true state being in event E. Then, they define "i knows event E at state t" to mean that i assigns posterior probability 1 to E at t. Accordingly, the event that i knows E, which is denoted by $K_i^{BD}(E)$, is then the set of t's such that i knows E at t. In this section, we derive the regular conditional probability and show that the knowledge function of BD is equivalent to the one we provide above.

Given the prior belief p on T and event $E \subseteq T$, let $p(E) = \sum_{t \in E} p(t)$ denote the agents' common prior belief that the true state is in event E. For any agent $i \in N$ and state $t \in T$, define $\mathcal{P}_i(t) = \{r \in T |\ r_i = t_i\}$ to be agent i's information set in state t. Note that $\mathcal{P}_i(t_i, t_{-i}) = \mathcal{P}_i(t_i, t'_{-i})$ for all $t_{-i}, t'_{-i} \in T_{-i}$. Therefore, for any given t_{-i}, $\mathcal{P}_i = \{\mathcal{P}_i(t_i, t_{-i}) |\ t_i \in T_i\}$ denotes agent i's **information partition**.

We next derive the conditional probability and the regular conditional probability. By Assumption 1 and Billingsley (1995, p. 428), for all $E \subseteq T$ and all $i \in N$ with her information partition $\mathcal{P}_i$, there exists a unique function $P_i(E|\mathcal{P}_i) : T \to \mathbb{R}$ that satisfies (i) $P_i(E|\mathcal{P}_i)$ is constant on $\mathcal{P}_i(t)$ for each t and (ii) for all $A \in \mathcal{P}_i$,

$\sum_{t \in A} P_i(E|\mathcal{P}_i)(t)p(t) = p(E \cap A)$. The function $P_i(E|\mathcal{P}_i)$ is defined as

$$P_i(E|\mathcal{P}_i)(t) = \frac{p(E \cap \mathcal{P}_i(t))}{p(\mathcal{P}_i(t))},$$

and called agent i's **conditional probability** of $E \subseteq T$ given $\mathcal{P}_i$. Define for each agent $i \in N$, a function $q_i : 2^T \times T \to [0, 1]$ as follows: for all $t \in T$ and $E \subseteq T$,

$$q_i(E, t) = P_i(E|\mathcal{P}_i)(t).$$

Since the conditional probability is uniquely defined (i.e., it has only one version), q_i is called agent i's **regular conditional probability** given $\mathcal{P}_i$ provided that it satisfies the following condition.

Proposition A-0. *For all $i \in N$ and $t \in T$, the function $q_i(\cdot, t)$ is a probability measure.*

Proof of Proposition A-0. Pick $i \in N$ and $t \in T$. It is easy to see that $q_i(E, t) \in [0, 1]$ for all $E \subseteq T$, $q_i(\emptyset, t) = P_i(\emptyset|\mathcal{P}_i)(t) = 0$ and $q_i(T, t) = P_i(T|\mathcal{P}_i)(t) = 1$. Moreover, since T is finite, $q_i(\cdot, t) = P_i(\cdot|\mathcal{P}_i)(t)$ is countably additive; that is, for all countable collections $\{E_k \subseteq T : k = 1, 2 \ldots\}$ of pairwise disjoint sets:

$$P_i\left(\bigcup_{k=1}^{\infty} E_k|\mathcal{P}_i\right)(t) = \sum_{k=1}^{\infty} P_i\left(E_k|\mathcal{P}_i\right)(t).$$

Therefore, $q_i(\cdot, t)$ is a probability measure. □

Now, we are ready to define the BD knowledge function. The value of the regular conditional probability $q_i(E, t)$ describes agent i's **posterior belief** at state t that the true state is in event E.

Definition A-0. *The* **BD knowledge function of agent** *i is a mapping $K_i^{BD} : 2^T \backslash \emptyset \to T$ defined as $K_i^{BD}(E) = \{t \in T | q_i(E, t) = 1\}$.*

Next, We show that the BD knowledge function is equivalent to the the knowledge function given in Definition 3.

Proposition A-1. *For all $i = 1, 2$ and all non-empty $E \subseteq T$, $K_i^{BD}(E) = K_i(E)$.*

Proof of Proposition A-1. Fix event $E \subseteq T$ and agent $i \in N$. First, we show that $K_i^{BD}(E) \subseteq K_i(E)$. For any $t \in K_i^{BD}(E)$,

$$q_i(E,t) = \frac{p(E \cap \mathcal{P}_i(t))}{p(\mathcal{P}_i(t))} = 1,$$

hence, $p(E^c \cap \mathcal{P}_i(t)) = 0$. This implies that $p(t') = 0$ for for any $t' \in E^c \cap \mathcal{P}_i(t)$. Recall that $t'_i = t_i$ for any $t' \in \mathcal{P}_i(t)$. It follows that, if $(t_i, r_{-i}) \notin E$, then $p(r_{-i}|t_i) = 0$ for any $r_{-i} \in T_{-i}$, and hence $t \in K_i(E)$.

It remains to show that $K_i(E) \subseteq K_i^{BD}(E)$. Fix $t \in K_i(E)$. By Lemma 1, $p(t_i, r_{-i}) = 0$ for all $(t_i, r_{-i}) \notin E$. This implies that $p(E \cap \mathcal{P}_i(t)) = 1$, and therefore

$$q_i(E,t) = \frac{p(E \cap \mathcal{P}_i(t))}{p(\mathcal{P}_i(t))} = 1.$$

Hence $t \in K_i^{BD}(E)$. □

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